APPLICATION OF MOMENT DISTRIBUTION TO PRISMATIC AND NON-PRISMATIC CIRCULAR ARCHED BENTS

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APPLICATION OF
MOMENT DISTRIBUTION TO PRISMATIC AND NON-PRISMATIC
CIRCULAR ARCHED BENTS

By

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and

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We wish to express our appreciation to Professor J. Sterling Kinney for his advice as to the preparation of this thesis.

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INTRODUCTION

OBJECT

This work is an extension of the investigation which was first begun by Cain[1] into the practicability of applying the principles of moment distribution to the solution of arched bents. Cain determined the fixed end moment, carry over factor, and absolute stiffness of prismatic parabolic arches of 25' span for various rise/span ratios. Hanson and Jones[2] continued the work by investigating various spans and found that for prismatic parabolic arches, a definite relationship existed between these factors and the span for a given rise/span ratio. Mascenik[3] did similar work on non-prismatic parabolic arches.

The object of this thesis was to investigate prismatic and non-prismatic circular arches, to determine the fixed end moment, carry over factor, and absolute stiffness for such arches having a rise/span ratio between 0.08 and 0.40, and to apply the moment distribution method of analysis to bents having either type of circular arch as their top member.

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[1] Cain, H.G., Application of Moment Distribution to Arched Bents, Thesis, R.P.I., February, 1947.

[2] Hanson, B.L., and Jones, H.C., Method of Application of Moment Distribution to Solution of Arched Bents, Thesis, R.P.I., May, 1948.

[3] Mascenik, John, Method of Application of Moment Distribution to the Solution of Arched Bents with a Varying Moment of Inertia, Thesis, R.P.I., May, 1949.

DEFINITIONS OF SYMBOLS

S - Length of span in feet.

$\widehat{L}$ - Total length of arch in feet.

r - Rise in feet - distance from springing line to crown.

H - Horizontal thrust in kips.

M^f_r - Fixed end moment in kip feet at right springing.

M^f_l - Fixed end moment in kip feet at left springing.

V - Vertical reaction in kips.

I_x - Cross sectional moment of inertia at any point along the arch.

I_c - Cross sectional moment of inertia at the crown of the arch.

θ - The angle between tangent to working line of arch, and a line parallel to springing line, equal to

The angle subtended by that portion of the arch between the crown and any point on the arch.

ds - Increment of length along the arch.

ΔH - Total horizontal deflection of a point.

ΔV - Total vertical deflection of a point.

θ - Total angular rotation of a point.

δ_{hh} - Horizontal deflection of a point due to the application of a unit horizontal force.

δ_{vv} - Vertical deflection of a point due to the application of a unit vertical force.

δ_{hv} - Horizontal deflection of a point due to the application of a unit vertical force.

δ_{vh} - Vertical deflection of a point due to the application of a unit horizontal force.

δ'_{hm} - Horizontal deflection of a point due to the application of a unit moment.

δ'_{vm} - - Vertical deflection of a point due to the application of a unit moment.

$\alpha_{\theta m}$ - Angular rotation of the member at a point due to the application of a unit moment.

$\alpha'_{\theta v}$ - Angular rotation of the member at a point due to the application of a unit vertical force.

$\alpha'_{\theta h}$ - Angular rotation of the member at a point due to the application of a unit horizontal force.

$\bar{y}$ - The vertical distance from the arch center to the neutral point.

X_{1p} - The horizontal distance from the vertical reference axis to the point of application of the unit load.

θ_{1p} - The angle subtended by the arc between crown and load point. ($\sin^{-1}X_{1p}/R$)

R - Radius of arch measured in feet.

H_o - Horizontal reaction at the neutral point.

V_o - Vertical reaction at the neutral point.

M_o - Moment at the neutral point.

L.P. - Load point - point of application of the load.

A - The angle subtended by the arc between the crown and the right or left springing point.

SIGN CONVENTION

The carry-over factor is considered positive when the moment applied at the hinged end produces tension on the top of the arch and the moment induced at the fixed end produces tension on the bottom of the arch.

Positive tabulated and plotted fixed end moments indicate tension on the top of the member at the end considered.

The moments considered in the Moment Distribution Method are all internal moments or in other words, the moment exerted by the member on the joint. If the end of the member tends to rotate the joint in a clockwise direction, the internal moment is considered to be positive.

PROCEDURE

PROCEDURE

Introduction

It is a well known fact that indeterminate rigid frames may be solved by the method of moment distribution provided the constants (a) stiffness at each end, (b) fixed-end moments at each end, and (c) carry-over factor at each end, are known for each member making up the frame.

Curves giving the required constants for non-prismatic members having one edge straight have been available to designers for some time; however, to the knowledge to the authors, no such complete information has been made available on prismatic and non-prismatic circular arched members.

In this thesis, the constants required for a solution by moment distribution have been determined for prismatic circular arched members, and for non-prismatic circular arched members having rise/span ratios of 0.04, 0.08, 0.20, 0.30, and 0.40.

The exact formula for each of the constants was derived in terms of the geometrical characteristics of the arch, and the relationship between these constants and the span for the same rise/span ratio was determined. The solution of the derived formulas for an arch having a span of one foot provided data for curves from which the required

constants for circular arched members of any span having rise/span ratios within the range of 0.04 to 0.40 can be determined.

The method of analysis used to derive the formula for fixed end moment is a combination of The Conjugate Structure Method and the Neutral Point Method. The General Method and the theory of Virtual Work were used to derive formulas for carry-over factor, absolute stiffness and the effect of spread.

The Conjugate Structure Method was developed by J. Sterling Kinney, Professor of Structural Engineering, R.P.I., Troy, New York. The principles of the method are as follows:

"(1) The conjugate structure, for a given real structure, is identical to the real structure with regard to the lengths of the members and their relative position.

(2) The conjugate structure is positioned in a horizontal plane.

(3) The load, which acts in a vertical direction on the conjugate structure, is the M/EI diagram of the real structure, that is, the conjugate structure is loaded with the flexural strains of the real structure.

(4) If the flexural strain at a given section of the real structure is such as to cause tension on the outside fibers, then this flexural strain is represented as a downward load on the conjugate structure. If compression exists on the outside fibers of the real structure, the load on the corresponding section of the conjugate structure is up.

(5) The conjugate structure, under the action of the real structure flexural strains as loads and the reactions of the conjugate structure, must satisfy three equilibrium condition equations, specifically:

$$\Sigma M_x = 0$$

$$\Sigma M_y = 0$$

$$\Sigma V = 0$$

(6) The shear at any section of the conjugate structure is the slope of the corresponding section of the real structure.

(7) The internal moment on any section of the conjugate structure is the deflection of the corresponding section of the real structure in a direction perpendicular to the lever arm used to find any particular moment.

(8) The end of the conjugate structure corresponding to the end of the real structure which suffers deflection always has a fixed support.

(9) If the moment at any section of the conjugate structure results in tension on the top fiber, the vertical deflection at the corresponding section of the real structure is down and the horizontal deflection is such as to shorten the horizontal projection of the distance between given points in the real structure.

(10) If a section be passed through any point of the conjugate structure and if the portion of the conjugate structure to the right of the section tends to move up,

with respect to the part on the left of the section, then the rotation of the corresponding section of the real structure is clockwise."[1]

It should be pointed out that in this thesis an elastic area, whose width at any point is 1/EI for the corresponding section of the real structure, was built up and loaded with an intensity equal to the Moment at the corresponding section on the cut back structure. The product of the load intensity M and the differential elastic area ds/EI gives the differential elastic load on length ds of the conjugate structure. This method provides for a better pictorial representation of the integration for the non-prismatic arch since we do not have to modify the load intensity by 1/EI.

FIXED END MOMENT AND THRUST DETERMINATION.

By locating the centroid (called the Neutral Point) of the elastic area, cantilevering the arch out from the right end, and connecting the Neutral Point (N.P.) to the left end by a weightless rigid bracket, we can obtain the deflections δ_{vv}, δ_{hh}, and α_{mm} as caused by applying individually a 1 kip vertical force, a 1 kip horizontal force, and a 1 foot kip couple at the N.P. We can also evaluate, by principles of the conjugate structure method, the deflections ΔH, ΔV and θ at the N.P. as caused by a 1 kip vertical load placed at any point on the arch.

From the theories of the Neutral Point Method, the reac-

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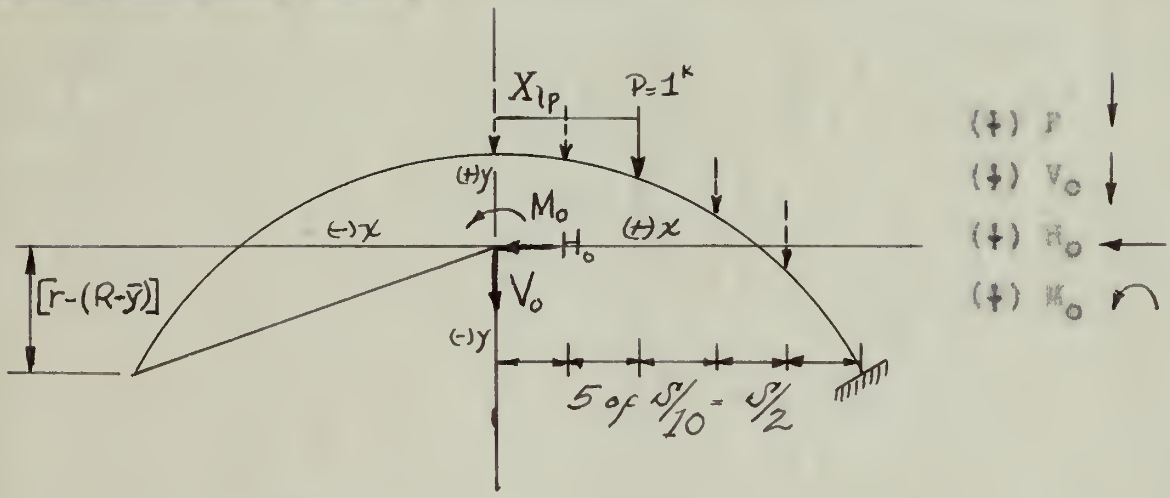
[1] "Indeterminate Structures" - J. Sterling Kinney, August 1949

tions at the N.P. caused by the real kip vertical load on the arch are

$$V_o = (-)\frac{\Delta V}{\delta vv} = V_L \qquad H_o = (-)\frac{\Delta H}{\delta hh} = H_L \qquad M_o = -\frac{\theta}{\alpha\,\theta m}$$

Should the value of a reaction come out negative, this indicates that the direction of the reaction is opposite to the direction of the corresponding unit force or couple applied at the N.P.

By applying the reactions at the N.P. and using the sign convention shown below:



$$M_l^f = V_o\left(\frac{S}{2}\right) - H_o\left[r - (R-\bar{y})\right] + M_o$$

$$M_r^f = V_o\left(\frac{S}{2}\right) - H_o\left[r - (R-\bar{y})\right] + M_o + P\left(\frac{S}{2} - X_{1p}\right)$$

A positive value of moment indicates tension on the top of the arch.

The value of $\bar{y}$ for a prismatic arch $= RS/\widehat{L}$.

The value of $\bar{y}$ for a non-prismatic arch $= \dfrac{\widehat{L}R + S(R-r)}{2S}$

CARRY-OVER FACTOR, ABSOLUTE STIFFNESS AND SPREAD

To avoid repetition, the procedure for the determination of these factors is explained on following pages along with the development of the formulas for their values.

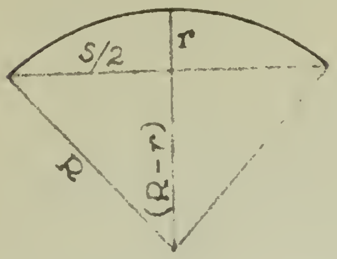
Sample calculations for the determination of the fixed end moments, vertical shear, horizontal reactions, carry-over factor, absolute stiffness and reactions due to spread are shown for prismatic arches in Appendix "A", and for non-prismatic arches in Appendix "B".

Tabulated results, with all values expressed in terms of the span length in feet, are shown on pages 56 and 57 for prismatic arches and on pages 65 and 66 for non-prismatic arches.

DEVELOPMENT OF METHOD FOR PRISMATIC ARCH

[illegible]

[illegible]



$$R^2 = (R-r)^2 + (S/2)^2$$

$$R^2 = R^2 - 2Rr + r^2 + \frac{S^2}{4}$$

$$2Rr = \frac{4r^2 + S^2}{4}$$

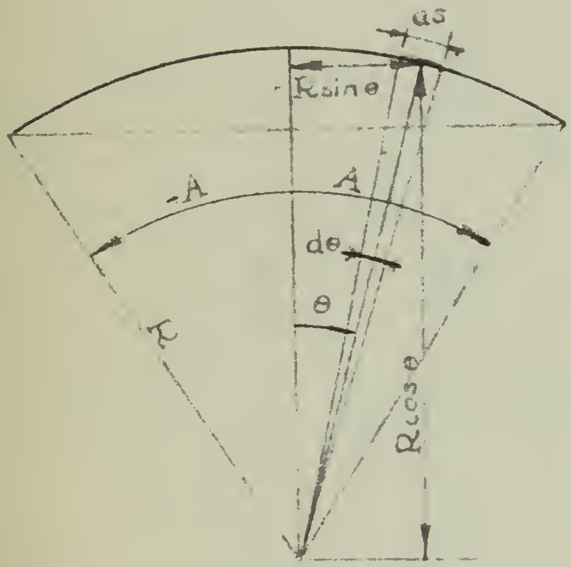
$$R = \frac{4r^2 + S^2}{8r}$$

[illegible]

$$\bar{x} = \frac{\int x \, [illegible]}{\int [illegible]} = [illegible]$$

$$\bar{y} = \frac{\int [illegible] \, ds}{\int ds} = [illegible]$$

$$= [illegible]$$



[illegible] symmetric [illegible]:

$$\bar{x} = 0$$

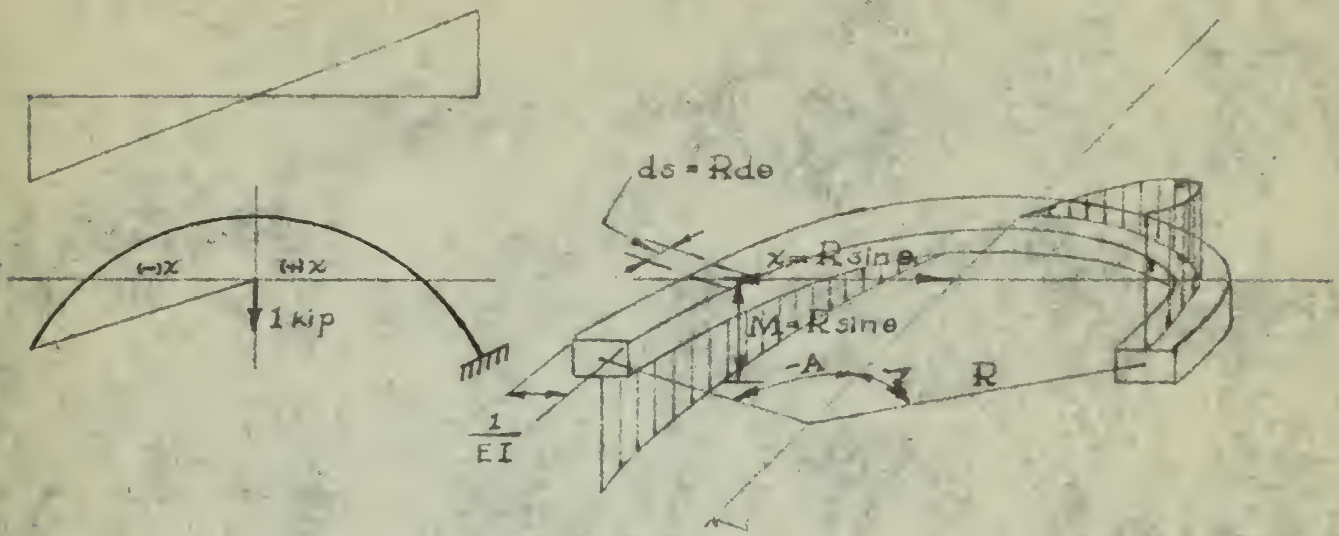
$$\bar{y} = [illegible]$$

III. DETERMINATION OF DEFLECTION OF NEUTRAL POINT DUE TO:

(A) 1 KIP VERTICAL FORCE AT N.P.

Moment Diagram

Loaded
Conjugate Structure



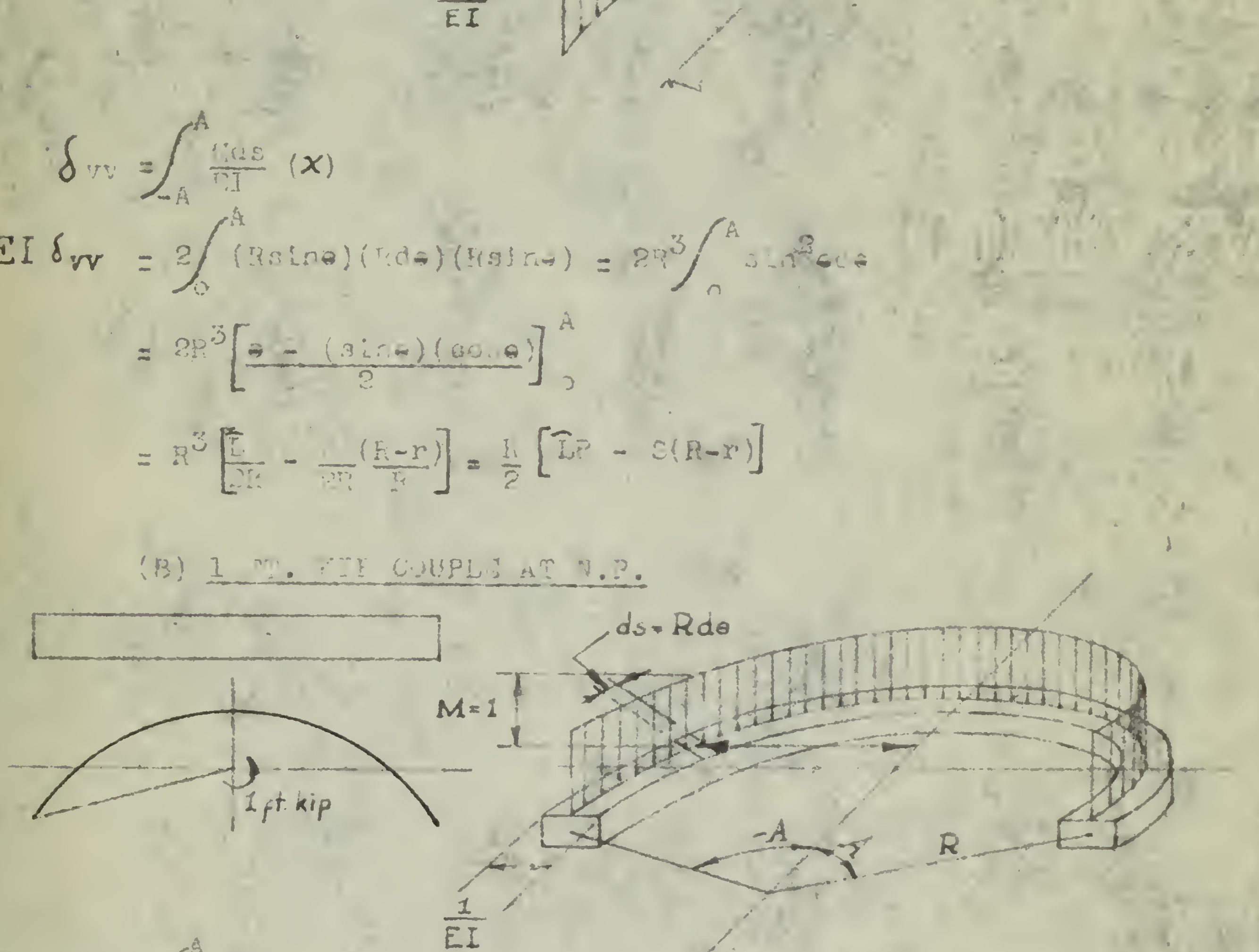
$$\delta_{vv} = \int_{-A}^{A} \frac{Mds}{EI}\,(x)$$

$$EI\,\delta_{vv} = 2\int_{0}^{A} (R\sin\theta)(Rd\theta)(R\sin\theta) = 2R^3\int_{0}^{A} \sin^2\theta\, d\theta$$

$$= 2R^3\left[\frac{\theta - (\sin\theta)(\cos\theta)}{2}\right]_{0}^{A}$$

$$= R^3\left[\frac{\widehat{L}}{2R} - \frac{C}{2R}\left(\frac{R-r}{R}\right)\right] = \frac{R}{2}\left[\widehat{L}R - C(R-r)\right]$$

(B) 1 FT. KIP COUPLE AT N.P.

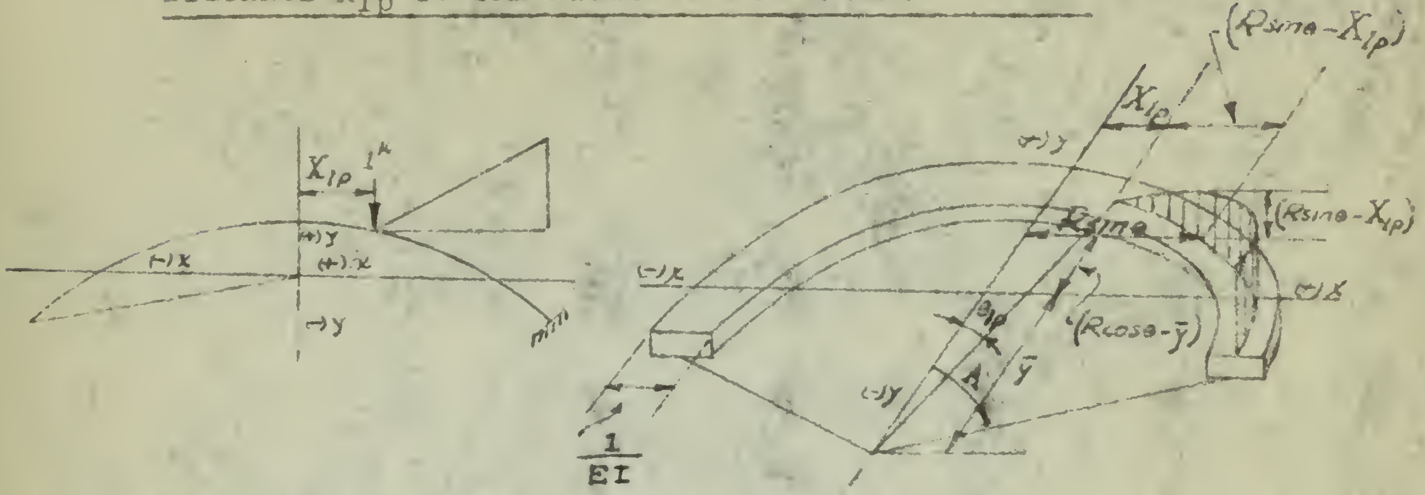


$$\alpha_{e} = \int_{-A}^{A} \frac{Mds}{EI}$$

$$EI\alpha_{em} = \int_{-A}^{A} (1)Rd\theta = 2\int_{0}^{A} Rd\theta = \widehat{L}$$

NOTE Pages 13 & 14 are interchanged.

IV. THE DETERMINATION OF DEFLECTIONS AT THE NEUTRAL POINT (N.P.) CAUSED BY A 1 KIP. VERTICAL LOAD ACTING ON THE ARCH AT A POINT THAT IS HORIZONTAL DISTANCE X_{1p} TO THE RIGHT OF THE CROWN.



A) DETERMINATION OF VERTICAL DEFLECTION AT N.P.

$$\Delta V = \int_{\theta_{1p}}^{A} \frac{Mds(x)}{EI}$$

$$EI\,\Delta V = \int_{\theta_{1p}}^{A} (R\sin\theta - X_{1p})(Rd\theta)(R\sin\theta)$$

$$= \int_{\theta_{1p}}^{A} R^3(\sin^2\theta)d\theta - \int_{\theta_{1p}}^{A} R^2(X_{1p})(\sin\theta)d\theta$$

$$= R^3\left[\frac{\theta - (\sin\theta)(\cos\theta)}{2}\right]_{\theta_{1p}}^{A} + R^2(X_{1p})\left[\cos\theta\right]_{\theta_{1p}}^{A}$$

$$= \frac{R^3}{2}\left[\frac{\overset{\frown}{L}}{2R} - \frac{S}{2R}\frac{(R-r)}{R}\right] - \frac{R^3}{2}\left[\theta_{1p} - (\sin\theta_{1p})(\cos\theta_{1p})\right] + R^2(X_{1p})\left[\frac{R-r}{R}\right]$$

$$- R^2(X_{1p})(\cos\theta_{1p})$$

$$= \frac{R}{4}\left[\overset{\frown}{L}R - S(R-r)\right] - \frac{R^3}{2}\left[\theta_{1p} - (\sin\theta_{1p})(\cos\theta_{1p})\right] + R(X_{1p})(R-r)$$

$$- R^2(X_{1p})(\cos\theta_{1p}).$$

(B) DETERMINATION OF HORIZONTAL DEFLECTION AT N.P.

$$\Delta H = \int_{\theta_{1p}}^{A} \frac{Mds(y)}{EI}$$

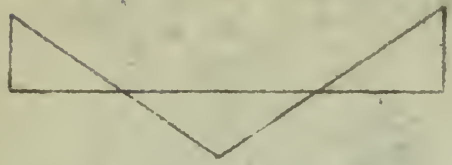
$$EI\,\Delta_H = \int_{\theta_{1p}}^{A} (R\sin\theta - X_{1p})\, Rd\theta\, (R\cos\theta - \bar{y})$$

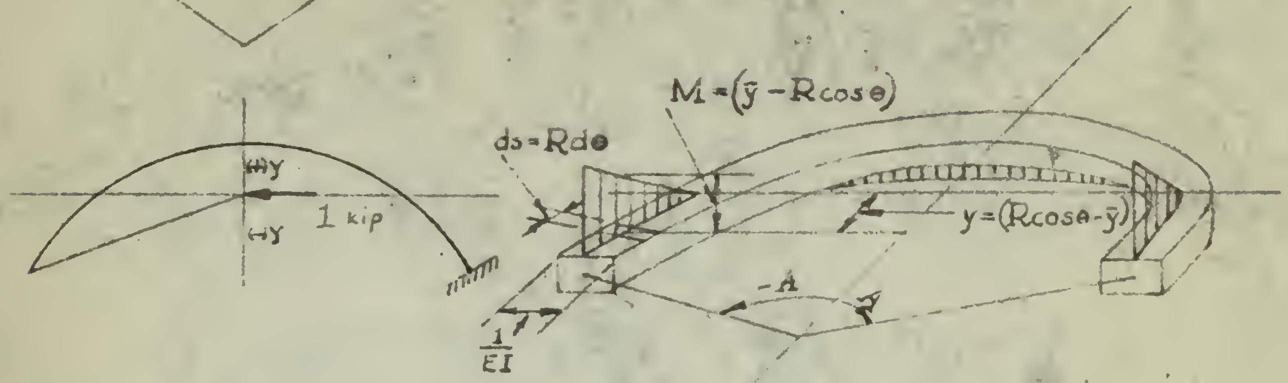
$$= \int_{\theta_{1p}}^{A} R^3(\sin\theta)(\cos\theta)d\theta - \int_{\theta_{1p}}^{A} R^2(\bar{y})(\sin\theta)d\theta - \int_{\theta_{1p}}^{A} R^2(X_{1p})(\cos\theta)d\theta$$

$$+ \int_{\theta_{1p}}^{A} R(\bar{y})(X_{1p})d\theta$$

13 NOTE PAGES 13 & 14 are INTERCHANGED

c) 1 KIP HORIZONTAL FORCE AT N.P.





$$\delta_{hh} = \int_{-A}^{A} \frac{M\,ds}{EI}(y)$$

$$EI\,\delta_{hh} = 2\int_0^A (\bar{y}-R\cos\theta)R\,d\theta(R\cos\theta-\bar{y})$$

$$= 2\int_0^A \left[(-)(\bar{y})^2+2(\bar{y})R\cos\theta-R^2\cos^2\theta\right] R\,d\theta$$

$$= 2\left[(-)R(\bar{y})^2\int_0^A d\theta + 2R^2(\bar{y})\int_0^A \cos\theta\,d\theta - R^3\int_0^A \cos^2\theta\,d\theta\right]$$

$$= 2\left[(-)R(\bar{y})^2\left[\theta\right]_0^A + 2R^2(\bar{y})\left[\sin\theta\right]_0^A - R^3\left(\frac{\theta+\sin\theta\cos\theta}{2}\right)\right]$$

$$= 2\left[(-)R\left(\frac{R^2S^2}{\widehat{L}^2}\right)\frac{\widehat{L}}{2R} + 2R^2\left(\frac{RS}{\widehat{L}}\right)\left(\frac{S}{2R}\right) - \frac{R^3}{2}\left[\frac{\widehat{L}}{2R} + \frac{S}{2R}\frac{(R-r)}{R}\right]\right]$$

$$= (-)\frac{R^2S^2}{\widehat{L}} + \frac{2R^2S^2}{\widehat{L}} - \frac{R}{2}\left[\widehat{L}R + S\,(R-r)\right]$$

$$= \frac{R^2S^2}{\widehat{L}} - \frac{R}{2}\left[\widehat{L}R + S\,(R-r)\right]$$

DETERMINATION OF HORIZONTAL DEFLECTION AT N.P. (Cont'd)

$$= R^3\left[\frac{\sin^2\theta}{2}\right]_{\theta_{1p}}^{A} + R^2(\bar{y})\left[\cos\theta\right]_{\theta_{1p}}^{A} - R^2(X_{1p})\left[\sin\theta\right]_{\theta_{1p}}^{A} + R(\bar{y})(X_{1p})\left[\theta\right]_{\theta_{1p}}^{A}$$

$$= \frac{R^3}{2}\left[\frac{S}{2R}\right]^2 - \frac{R^3}{2}(\sin^2\theta_{1p}) + R^2\left(\frac{RS}{\widehat{L}}\right)\left(\frac{R-r}{R}\right) - R^2\left(\frac{RS}{\widehat{L}}\right)\cos\theta_{1p} - R^2(X_{1p})\left(\frac{S}{2R}\right)$$

$$+ R^2(X_{1p})(\sin\theta_{1p}) + R\left(\frac{RS}{\widehat{L}}\right)(X_{1p})\left(\frac{\widehat{L}}{2R}\right) - R\left(\frac{RS}{\widehat{L}}\right)(X_{1p})\theta_{1p}$$

$$= \frac{RS^2}{8} - \frac{R^3}{2}(\sin^2\theta_{1p}) + \frac{R^2S}{\widehat{L}}(R-r) - \frac{R^3S}{\widehat{L}}(\cos\theta_{1p}) - \frac{R(X_{1p})S}{2}$$

$$+ R^2(X_{1p})(\sin\theta_{1p}) + \frac{RS}{2}(X_{1p}) - \frac{R^2S}{\widehat{L}}(X_{1p})\theta_{1p}$$

$$= \frac{RS^2}{8} - \frac{R^3}{2}\left(\frac{X_{1p}}{R}\right)^2 + \frac{R^2S}{\widehat{L}}(R-r) - \frac{R^3S}{\widehat{L}}(\cos\theta_{1p}) + R^2(X_{1p})\left(\frac{X_{1p}}{R}\right) - \frac{R^2S}{\widehat{L}}(X_{1p})\theta_{1p}$$

$$= \frac{RS^2}{8} - \frac{R(X_{1p})^2}{2} + \frac{R^2S}{\widehat{L}}(R-r) - \frac{R^3S}{\widehat{L}}(\cos\theta_{1p}) + R(X_{1p})^2 - \frac{R^2S}{\widehat{L}}(X_{1p})\theta_{1p}$$

$$\Delta H = \frac{RS^2}{8} + \frac{R(X_{1p})^2}{2} + \frac{R^2S}{\widehat{L}}(R-r) - \frac{R^3S}{\widehat{L}}(\cos\theta_{1p}) - \frac{R^2S}{\widehat{L}}(X_{1p})\theta_{1p}$$

(C) DETERMINATION OF ANGULAR DEFLECTION AT N.P.

$$\theta = \int_{\theta_{1p}}^{A} \frac{Mds}{EI}$$

$$= \int_{\theta_{1p}}^{A} (R\sin\theta - X_{1p})Rd\theta$$

$$= \int_{\theta_{1p}}^{A} R^2(\sin\theta)d\theta - \int_{\theta_{1p}}^{A} R(X_{1p})d\theta$$

$$= (-)R^2\left[\cos\theta\right]_{\theta_{1p}}^{A} - R(X_{1p})\left[\theta\right]_{\theta_{1p}}^{A}$$

$$= -R^2\left(\frac{R-r}{R}\right) + R^2(\cos\theta_{1p}) - R(X_{1p})\left(\frac{\widehat{L}}{2R}\right) + R(X_{1p})\theta_{1p}$$

$$\theta = -R(R-r) + R^2(\cos\theta_{1p}) - \frac{(X_{1p})\widehat{L}}{2} + R(X_{1p})\theta_{1p}$$

(D) THE DETERMINATION OF THE REACTIONS AT E.P.

$$V_o = (-)\frac{\Delta V}{\delta_{vv}} \quad , \quad H_o = (-)\frac{\Delta H}{\delta_{hh}} \quad , \quad M_o = -\frac{\theta}{\alpha_{mm}}$$

$$V_o = (-)\frac{\frac{R}{4}\left[\widehat{L}R - S(R-r)\right] - \frac{R^3}{2}\left[\theta_{1p} - (\sin\theta_{1p})(\cos\theta_{1p})\right] + R(X_{1p})(R-r) - R^2(X_{1p})(\cos\theta_{1p})}{\frac{R}{2}\left[\widehat{L}R - S(R-r)\right]}$$

$$H_o = (-)\frac{\frac{RS^2}{3} + \frac{R(X_{1p})^2}{2} + \frac{R^2S}{\widehat{L}}(R-r) - \frac{R^3S}{\widehat{L}}(\cos\theta_{1p}) - \frac{R^2S}{\widehat{L}}(X_{1p})\theta_{1p}}{\frac{R^2S^2}{\widehat{L}} - \frac{R}{2}\left[\widehat{L}R + S(R-r)\right]}$$

$$M_o = (-)\frac{(-)R(R-r) + R^2(\cos\theta_{1p}) - \frac{(X_{1p})\widehat{L}}{2} + R(X_{1p})\theta_{1p}}{\widehat{L}}$$

(E) THE DETERMINATION OF THE FIXED END MOMENTS

Considering the assumed directions of the reactions at the E.P. on Page 9.

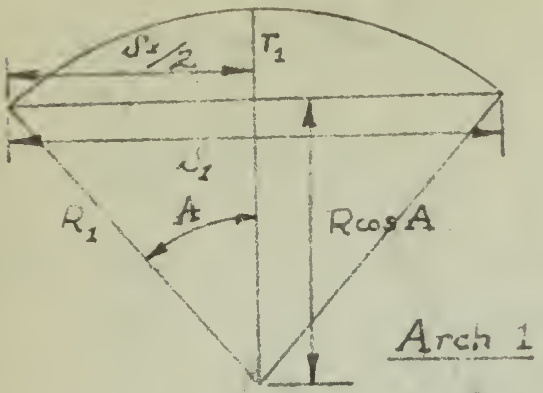
$$M_1^f = (+)V_o\left(\frac{S}{2}\right) - H_o\left[r - (R-\bar{y})\right] + M_o$$

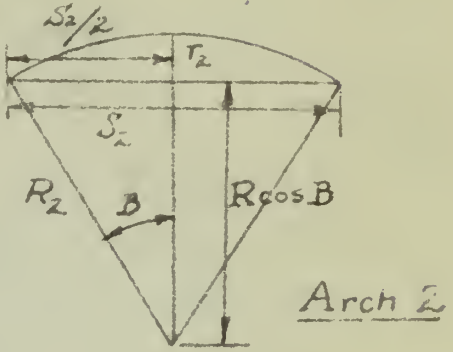
$$(+)V_o\left(\frac{S}{2}\right) - H_o\left[r - \left(R-\frac{RS}{\widehat{L}}\right)\right] + M_o$$

M_r^f is determined by writing moments about the right end. A positive value of moment indicates tension on the top of the arch.

V. DETERMINATION OF RELATIONSHIP BETWEEN MOMENT AND SPAN FOR THE SAME $\frac{\text{RISE}}{\text{SPAN}}$ RATIO.

Proof that arches of same $\frac{\text{Rise}}{\text{Span}}$ ratio are similar.





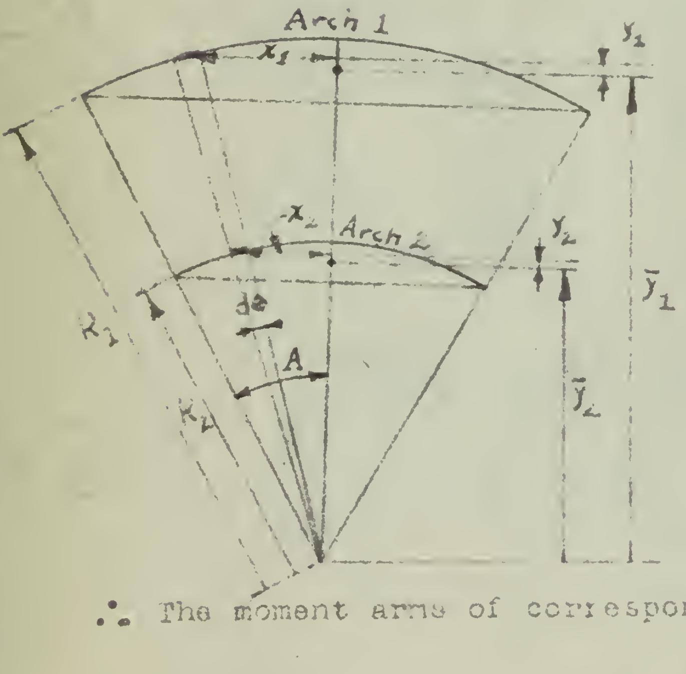
$r_1 = R_1 - R_1\cos A = R_1(1-\cos A)$ $\qquad$ $r_2 = R_2 - R_2\cos B = R_2(1-\cos B)$

$\frac{S_1}{2} = R_1 \sin A$ $\qquad$ $\frac{S_2}{2} = R_2 \sin B$

$\frac{r_1}{S_1/2} = \frac{1-\cos A}{\sin A} = \frac{\text{Rise}}{\text{Span}} =$ $\qquad$ $\frac{r_2}{S_2/2} = \frac{1-\cos B}{\sin B}$

$$\frac{1-\cos A}{\sin A} = \frac{1-\cos B}{\sin B}$$

$\therefore A = B$ AND THE ARCHES ARE SIMILAR.



$\frac{X_2}{X_1} = \frac{R_2}{R_1} = \frac{S_2}{S_1} \quad \therefore X_2 = X_1\frac{S_2}{S_1}$

$\frac{\bar{y}_2}{y_1} = \frac{\frac{R_2 S_2}{R_2\ A}}{\frac{R_1 S_1}{R_1\ A}} = \frac{S_2}{S_1}$

$\frac{\bar{y}_2 + y_2}{\bar{y}_1 + y_1} = \frac{S_2}{S_1}$

$\therefore \frac{y_2}{y_1} = \frac{S_2}{S_1}$ or $y_2 = y_1\frac{S_2}{S_1}$

$\therefore$ The moment arms of corresponding $\frac{Mds}{EI}$ loads vary as $\frac{S_2}{S_1}$

From Page 14 it is seen that M at any point due to a 1^k load on the Arch = $(R\sin\theta - X_{1p})$

$\therefore$ For Arch 2, $M_2 = R_2\sin\theta - R_2\sin\theta_{1p} = R_2(\sin\theta - \sin\theta_{1p})$

For Arch 1, $M_1 = R_1\sin\theta - R_1\sin\theta_{1p} = R_1(\sin\theta - \sin\theta_{1p})$

$$\therefore \quad \frac{M_2}{M_1} = \frac{R_2}{R_1} = \frac{S_2}{S_1} \quad \text{or} \quad M_2 = M_1\left(\frac{S_2}{S_1}\right)$$

Note: The above M's are moment intensities acting on areas $\frac{ds}{EI}$, or the moment existing at the corresponding section of the cut back structure. $ds_2 = \frac{R_2}{R_1} ds_1 = \frac{S_2}{S_1} ds_1$

The Elastic Loads $\frac{Mds}{EI}$ vary as $\left(\frac{S_2}{S_1}\right)^2$

Since: $(\Delta V)_1 = \int_{\theta_{1p}}^{A} \left(\frac{Mds}{EI}\right)_1 (X_1)$ and $(\Delta V)_2 = \int_{\theta_{1p}}^{A} \left(\frac{Mds}{EI}\right)_2 (X_2)$

$$\frac{(\Delta V)_2}{(\Delta V)_1} = \frac{\int_{\theta_{1p}}^{A} \left(\frac{Mds}{EI}\right)_1 \left(\frac{S_2}{S_1}\right)^2 (X_1)\left(\frac{S_2}{S_1}\right)}{\int_{\theta_{1p}}^{A} \left(\frac{Mds}{EI}\right)_1 (X_1)} = \left(\frac{S_2}{S_1}\right)^3$$

By the same reasoning $(\delta_{vv})_2$ as caused by 1 kip vertical at $(N.P.)_2 = (\delta_{vv})_1\left(\frac{S_2}{S_1}\right)^3$

$$\text{Hence } \frac{(V_o)_2}{(V_o)_1} = \frac{(-)\frac{(\Delta V)_2}{(\delta vv)_2}}{(-)\frac{(\Delta V)_1}{(\delta vv)_1}} = \frac{\frac{(\Delta V)_1 (S_2/S_1)^3}{(\delta vv)_1 \left(\frac{S_2}{S_1}\right)^3}}{\frac{(\Delta V)_1}{(\delta vv)_1}} = 1 \text{ or } (V_o)_2 = (V_o)_1$$

By similar reasoning $(H_o)_2 = (H_o)_1$

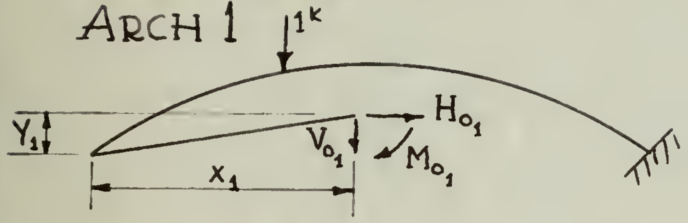
Thus under the action of similarly placed equal loads, the vertical and horizontal reactions at $(N.P.)_1$ are the same as those at $(N.P.)_2$.

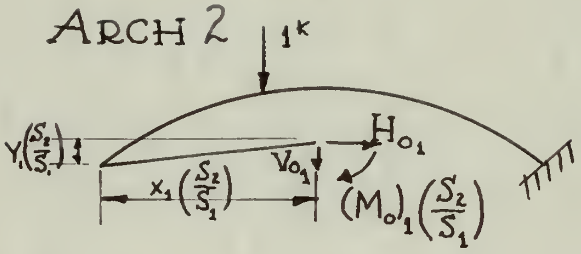
$$Mo_2 = (-)\frac{\theta_2}{\alpha_2} = (-)\frac{\int_{\theta_{1p}}^{A}(\frac{Mds}{EI})_2}{\widehat{L}_2} \quad , \quad Mo_1 = (-)\frac{\theta_1}{\alpha_1} = (-)\frac{\int_{\theta_{1p}}^{A}(\frac{Mds}{EI})_1}{\widehat{L}_1}$$

$$\therefore \quad \frac{(Mo)_2}{(Mo)_1} = \frac{\dfrac{\int_{\theta_{1p}}^{A}(\frac{Mds}{EI})_1 \cdot (\frac{S_2}{S_1})^2}{\widehat{L}_1 \cdot (\frac{S_2}{S_1})}}{\dfrac{\int_{\theta_{1p}}^{A}(\frac{Mds}{EI})_1}{\widehat{L}_1}} = \frac{S_2}{S_1}$$

$(Mo)_2 = (Mo_1)\frac{S_2}{S_1}$ (Mo's vary directly as Span)

Since the Moment Arms of $(V_o)_2$ and $(H_o)_2$ to any point in Arch 2 are $\frac{S_2}{S_1}$ times the moment arms of $(V_o)_1$ and $(H_o)_1$ to the corresponding point in Arch 1, and since $(M_o)_2 = (M_o)_1(\frac{S_2}{S_1})$, then it is proved that the fixed end moment of Arch 2 is equal to the fixed end moment of Arch 1 times $\frac{S_2}{S_1}$ as illustrated below.





Taking moments at left springing

$(M_L)_1 = (M_o)_1 + (V_{o1})x_1 + (H_o)_1 y_1 \qquad (M_L)_2 = (M_o)_1(\frac{S_2}{S_1}) + (V_{o1})x_1(\frac{S_2}{S_1}) + (H_o)_1 y_1(\frac{S_2}{S_1})$

$$\frac{(M_L)_2}{(M_L)_1} = \frac{S_2}{S_1} \quad \text{or} \quad (M_L)_2 = (M_L)_1\frac{S_2}{S_1}$$

or Fixed End Moment varies directly as the Span.

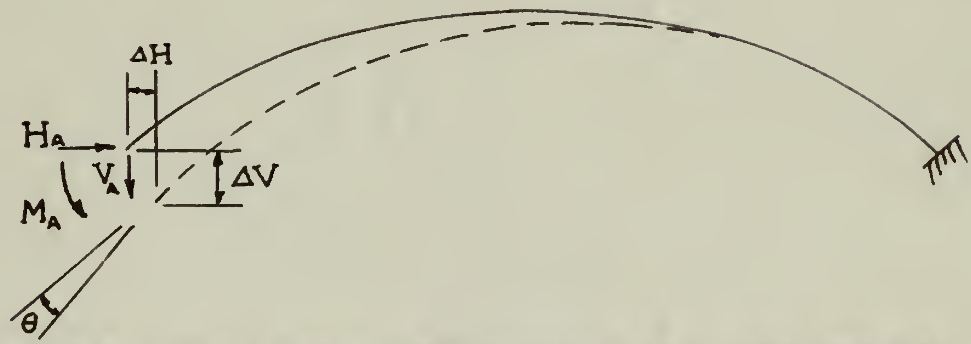
VI. DETERMINATION OF CARRY-OVER FACTOR

When an arch is cut at the left springing and redundant forces are applied as shown below, deflections of the left springing will be as follows:

(1) $\Delta V = M_a \delta'_{vm} + V_a \delta_{vv} + H_a \delta_{vh}$

(2) $\Delta H = M_a \delta'_{hm} + V_a \delta_{hv} + H_a \delta_{hh}$

(3) $\theta = M_a \alpha_{\theta m} + V_a \alpha'_{\theta v} + H_a \alpha'_{\theta h}$



By multiplying both sides of equations (1), (2) and (3) by EI, these equations take the form:

(4) $EI\,\Delta V = M_a(EI\,\delta'_{vm}) + V_a(EI\,\delta_{vv}) + H_a(EI\,\delta_{vh})$

(5) $EI\,\Delta H = M_a(EI\,\delta'_{hm}) + V_a(EI\,\delta_{hv}) + H_a(EI\,\delta_{hh})$

(6) $EI\theta = M_a(EI\,\alpha_{\theta m}) + V_a(EI\,\alpha'_{\theta v}) + H_a(EI\,\alpha'_{\theta h})$

From the theories of virtual work and Maxwell's Law of Reciprocal Deflections:

$$EI\,\delta'_{vm} = EI\,\alpha'_{\theta v} = \int x\,ds = \widehat{L} \cdot \frac{S}{2}$$

$$EI\,\delta'_{hm} = EI\,\alpha'_{\theta h} = \int y\,ds = \widehat{L} \cdot \left[\text{Rise}-(R-\bar{y})\right]$$

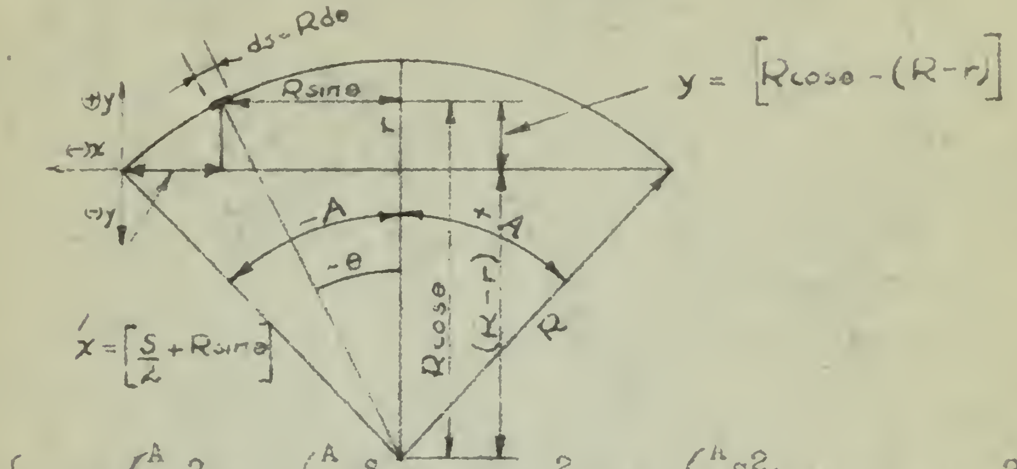
$$EI\,\alpha_{\theta m} = \int ds = \widehat{L}$$

$$EI\,\delta_{vh} = EI\,\delta_{hv} = \int xy\,ds = \widehat{L} \cdot \frac{S}{2} \cdot \left[\text{Rise}-(R-\bar{y})\right]$$

$$EI\,\delta_{vv} = \int X^2 ds$$

$$EI\,\delta_{hh} = \int y^2 ds$$

From the figure below, the right sides of the above equations are integrated such that $EI\delta_{vv}$ and $EI\delta_{hh}$ are expressed in terms of Rise, Span, Radius and Arch length.



$$EI\,\delta_{vv} = \int_{-A}^{A} X^2 ds = \int_{-A}^{A} \left(\frac{S}{2} + R\sin\theta\right)^2 R\,d\theta = \int_{-A}^{A} \left(\frac{S^2}{4} + SR\sin\theta + R^2\sin^2\theta\right) R\,d\theta$$

$$= \frac{RS^2}{4}\int_{-A}^{A} d\theta + SR^2\int_{-A}^{A} \sin\theta\,d\theta + R^3\int_{-A}^{A} \sin^2\theta\,d\theta$$

$$= \frac{RS^2}{4}\Big[\theta\Big]_{-A}^{A} - SR^2\Big[\cos\theta\Big]_{-A}^{A} + R^3\left[\frac{\theta - \sin\theta\cos\theta}{2}\right]_{-A}^{A}$$

$$= \frac{RS^2}{4}(2)\frac{\widehat{L}}{2R} - 0 + \frac{R^3}{2}\left[\frac{\widehat{L}}{2R} - \frac{S}{2R}\left(\frac{R-r}{R}\right) + \frac{\widehat{L}}{2R} - \frac{S}{2R}\left(\frac{R-r}{R}\right)\right]$$

$$= \frac{S^2\widehat{L}}{4} + \frac{R}{2}\Big[\widehat{L}R - S(R-r)\Big]$$

$$EI\,\delta_{hh} = \int_{-A}^{A} y^2 ds = 2\int_{0}^{A}\Big[R\cos\theta - (R-r)\Big]^2 R\,d\theta = 2\int_{0}^{A}\Big[R^2\cos^2\theta - 2(R-r)R\cos\theta +$$

$$+ (R-r)^2\Big]R\,d\theta$$

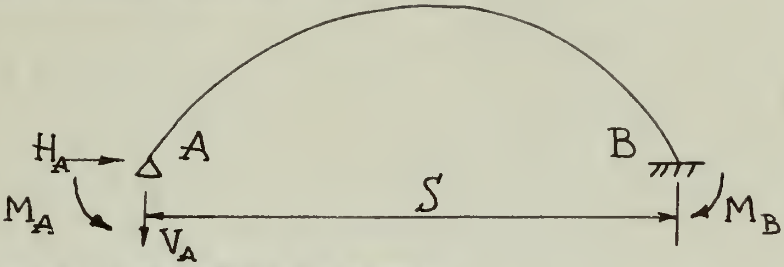
$$= 2\int_{0}^{A} R^3\cos^2\theta\,d\theta - 4\int_{0}^{A} R^2(R-r)\cos\theta\,d\theta + 2\int_{0}^{A} R(R-r)^2 d\theta$$

$$= 2R^3\left[\frac{\theta + \sin\theta\cos\theta}{2}\right]_{0}^{A} - 4R^2(R-r)\Big[\sin\theta\Big]_{0}^{A} + 2R(R-r)^2\Big[\theta\Big]_{0}^{A}$$

$$= R^3\left[\frac{\widehat{L}}{2R} + \frac{S}{2R}\frac{(R-r)}{R}\right] - 4R^2(R-r)\frac{S}{2R} + 2R(R-r)^2\,\frac{\widehat{L}}{2R}$$

$$= \frac{R}{2}\Big[\widehat{L}R + S(R-r)\Big] - 2R(R-r)S + (R-r)^2\widehat{L}$$

Since the carry-over factor is equal numerically to the moment induced at the fixed end of a member when a unit moment is applied at the hinged end, the carry-over factor for the arch may be obtained by equating equations (4) and (5) to zero and solving simultaneously for V_a and H_a in terms of M_a. Then by taking moments about the fixed end, M_b can be expressed as the carry-over factor times M_a.



Equating (4) and (5) to zero

$$(4')\ 0 = EI\left[M_a(\delta'_{vm}) + V_a(\delta_{vv}) + H_a(\delta_{vh})\right]$$

$$(5')\ 0 = EI\left[M_a(\delta'_{hm}) + V_a(\delta_{hv}) + H_a(\delta_{hh})\right]$$

Multiplying (4') by δ_{hh} and (5') by $(-)\ \delta_{vh}$, and dividing (4') and (5') by EI

$$(-)V_a(\delta_{vv})(\delta_{hh}) = M_a(\delta'_{vm})(\delta_{hh}) + H_a\delta_{vh}\delta_{hh}$$

$$(+)V_a(\delta_{hv})(\delta_{vh}) = (-)M_a(\delta'_{hm})(\delta_{vh})(-)\ H_a\delta_{hh}\delta_{vh}$$

$$V_a = \frac{(EI)^2}{(EI)^2}\left[\frac{(\delta'_{vm})(\delta_{hh}) - (\delta'_{hm})(\delta_{vh})}{(\delta_{hv})(\delta_{vh}) - (\delta_{vv})(\delta_{hh})}\right]M_a$$

For sake of brevity, we write the above equation as $V_a = (D_2)M_a$

Multiplying (4') by δ_{hv} and (5') by $(-)\ \delta_{vv}$

$$(-)\ H_a(\delta_{vh})(\delta_{hv}) = M_a(\delta'_{vm})(\delta_{hv}) + V_a(\delta_{vv})(\delta_{hv})$$

$$(+)\ H_a(\delta_{hh})(\delta_{vv}) = (-)M_a(\delta'_{hm})(\delta_{vv})\ (-)\ V_a(\delta_{hv})(\delta_{vv})$$

$$H_a = \frac{(EI)^2}{(EI)^2}\left[\frac{(\delta'_{vm})(\delta_{hv}) - (\delta'_{hm})(\delta_{vv})}{(\delta_{hh})(\delta_{vv}) - (\delta_{vh})(\delta_{hv})}\right]M_a$$

We write the above as: $H_a = (D_1)M_a$

Taking moments at the fixed end

$$M_b = M_a + V_a S$$

$$= M_a + (D_2) M_a S = (1 + D_2 S)(M_a)$$

The induced moment at the fixed end

$$-M_b = (-)(1 + D_2)(S) M_a$$

$$\underline{\text{The Carry-Over Factor}} = \underline{(-)(1 + D_2)(S)}$$

Although the term H_a does not appear in the moment equation above, this thrust must be considered in the solution of an Arched Bent by moment distribution since every time a joint is rotated and the moments balanced out at the joint, the horizontal restraining force at the ends of the arch resulting from the applied moments must be determined. By keeping a record of the applied moments, the H.R.F. may be determined by $H_a = (D_1) M_a$.

VI.(a) DETERMINATION OF RELATIONSHIP BETWEEN CARRY-OVER FACTOR AND SPAN FOR THE SAME RISE/SPAN RATIO.

Since the deflection condition equations (1), (2) and (3) are applicable to an arch of any span, it is necessary only to determine how the various deflections vary with span in order to find the relationship between carry-over factor and span.

From the fact that arches of the same Rise to Span ratio are similar as previously proved, the deflections for an arch of Span S_2 are written in terms of the deflections of an arch of Span S_1.

General	For Arch #1	For Arch #2
$EI\delta'_{vm} = EI\alpha'_{\theta v} = \int x\,ds$	$\int (X_1)\,ds_1$	$\int (X_1 \cdot \frac{S_2}{S_1})\,ds_1 \cdot \frac{S_2}{S_1} = (\frac{S_2}{S_1})^2 \int (X_1)\,ds_1$
$EI\delta'_{hm} = EI\alpha'_{\theta h} = \int y\,ds$	$\int (Y_1)\,ds_1$	$\int (Y_1 \cdot \frac{S_2}{S_1})\,ds_1 \cdot \frac{S_2}{S_1} = (\frac{S_2}{S_1})^2 \int (Y_1)\,ds_1$
$EI\alpha_{\theta m} = \int ds$	$\int ds_1$	$\int ds_1 \cdot \frac{S_2}{S_1} = (\frac{S_2}{S_1}) \int ds_1$
$EI\delta_{vh} = EI\delta_{hv} = \int xy\,ds$	$\int X_1 Y_1\,ds_1$	$\int (X_1 \cdot \frac{S_2}{S_1})(Y_1 \cdot \frac{S_2}{S_1})\,ds_1 \cdot \frac{S_2}{S_1} = (\frac{S_2}{S_1})^3 \int ds_1$
$EI\delta_{vv} = \int x^2\,ds$	$\int (X_1)^2\,ds_1$	$\int (X_1 \cdot \frac{S_2}{S_1})^2\,ds_1 \cdot \frac{S_2}{S_1} = (\frac{S_2}{S_1})^3 \int (X_1)^2\,ds_1$
$EI\delta_{hh} = \int y^2\,ds$	$\int (Y_1)^2\,ds_1$	$\int (Y_1 \cdot \frac{S_2}{S_1})^2\,ds_1 \cdot \frac{S_2}{S_1} = (\frac{S_2}{S_1})^3 \int (Y_1)^2\,ds_1$

Substituting the above deflections for Point A of Arch 2, for the corresponding deflections for Arch 1, we obtain

$$V_a = \frac{D_2 (\frac{S_2}{S_1})^5}{(\frac{S_2}{S_1})^6} M_a = D_2 (\frac{S_1}{S_2}) M_a$$

$$H_a = D_1 \frac{(\frac{S_2}{S_1})^5}{(\frac{S_2}{S_1})^6} M_a = D_1 (\frac{S_1}{S_2}) M_a$$

Taking Moments at the fixed end of Arch 2 of Span $S_2 = S_1(\frac{S_2}{S_1})$

$$M_b = M_a + V_a S_2 = M_a + D_2 (\frac{S_1}{S_2}) M_a S_2 = M_a + D_2 (\frac{S_1}{S_2}) M_a S_1 (\frac{S_2}{S_1})$$

$$M_b = M_a + D_2 M_a S_1 = (1 + D_2 S_1) M_a$$

The induced moment at the fixed end

$$-M_b = -(1 + D_2 S_1) M_a$$

The Carry-Over Factor for Arch 2 = $-(1 + D_2 S_1)$ which is the same as the C.O.F. for Arch 1.

$\therefore$ Carry-Over Factor for Arches of same Rise/Span ratio is Independent of Span.

VII DETERMINATION OF ABSOLUTE STIFFNESS

Since the absolute stiffness is defined as the moment required to produce unit rotation (1 radian) of the hinged end of a member whose far end is rigidly fixed, the absolute stiffness for the arch may be obtained by setting θ of equation (6) equal to 1, and substituting the values of V_a and H_a in terms of M_a (previously determined) in the right side of the equation. By solving the resulting equation for M_a, the absolute stiffness is determined.

$$(6) \quad EI(\theta) = M_a(EI\alpha_{\theta m}) + V_a(EI\alpha'_{\theta v}) + H_a(EI\alpha'_{\theta h})$$

After substitution:

$$EI(1) = M_a(EI\alpha_{\theta m}) + (D_2)M_a(EI\alpha'_{\theta v}) + (D_1)M_a(EI\alpha'_{\theta h})$$

$$M_a = \frac{EI}{EI\alpha_{\theta m} + (D_2)EI\alpha'_{\theta v} + (D_1)EI\alpha'_{\theta h}}$$

which, for sake of brevity, is written

$$M_a = \frac{EI}{D_3} = \text{Absolute Stiffness}$$

VIIA DETERMINATION OF RELATIONSHIP BETWEEN ABSOLUTE STIFFNESS AND SPAN FOR THE SAME RISE/SPAN RATIO.

Using the relationship between deflections and span that have been previously determined, we write equation (6) for an arch of Span S_2

$$EI(\theta) = M_a\left[EI(\alpha_{\theta m}\frac{S_2}{S_1})\right] + V_a\left[EI(\alpha'_{\theta v})(\frac{S_2}{S_1})^2\right] + H_a\left[EI(\alpha'_{\theta h})(\frac{S_2}{S_1})^2\right]$$

Substituting the values of V_a and H_a as previously determined for an arch of span S_2

$$EI(1) = M_a\left[EI(\alpha_{\theta m}\frac{S_2}{S_1})\right] + D_2(\frac{S_1}{S_2})M_a\left[EI\alpha'_{\theta v}(\frac{S_2}{S_1})^2\right] + D_1(\frac{S_1}{S_2})M_a\left[EI\alpha'_{\theta h}(\frac{S_2}{S_1})^2\right]$$

$$EI(1) = M_a\left[EI\alpha_{\theta m}(\frac{S_2}{S_1})\right] + D_2M_a\left[EI\alpha'_{\theta v}(\frac{S_2}{S_1})\right] + D_1M_a\left[EI\alpha'_{\theta h}(\frac{S_2}{S_1})\right]$$

$$M_a = \frac{EI}{\left[EI\alpha_{\theta m} + (D_2)EI\alpha'_{\theta v} + D_1EI\alpha'_{\theta h}\right]\frac{S_2}{S_1}}$$

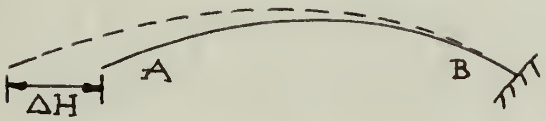
which is written $M_a = \frac{EI}{D_3}(\frac{S_1}{S_2})$

$\therefore$ Absolute Stiffness varies inversely as the Span.

VIII DETERMINATION OF THE INDUCED MOMENT AND REACTIONS (VERTICAL AND HORIZONTAL) OCCURRING AT THE END OF AN ARCH DUE TO SPREADING OF THE ARCH UNDER LOAD.

Every vertical load placed on an arch produces a tendency for the arch to spread. Since the column members of an arched bent do not have infinite stiffness, a definite spreading of the arch will occur under the load. This spread must be taken into consideration when solving an an arched bent by moment distribution.

To determine the reactions in terms of spread, we subject the left end of the arch to a spread of ΔH feet (allowing no vertical or rotational deflection of that end)



By equating equations (4) and (6) to zero and solving equations (4),(5) and (6) simultaneously in terms of EI ΔH,

we obtain the desired reactions.

$$\text{(a)}\quad 0 = M_a(EI\,\delta'_{vm}) + V_a(EI\,\delta_{vv}) + H_a(EI\,\delta_{vh})$$

$$\text{(b)}\quad 0 = M_a(EI\alpha_{\theta m}) + V_a(EI\alpha'_{\theta v}) + H_a(EI\alpha'_{\theta h})$$

Determination of V_a as caused by spread ΔH

By multiplying the above equations by $\alpha'_{\theta h}$ and $(-)\,\delta_{vh}$ respectively.

$$-V_a(EI\delta_{vv})\alpha'_{\theta h} = M_a(EI\delta'_{vm})\alpha'_{\theta h} + H_a(EI\delta_{vh})\alpha'_{\theta h}$$

$$V_a(EI\alpha'_{\theta v})\delta_{vh} = (-)M_a(EI\alpha_{\theta m})\delta_{vh}\ (-)\ H_a(EI\alpha'_{\theta h})\delta_{vh}$$

$$V_a = \frac{(EI)^2}{(EI)^2}\left[\frac{(\delta'_{vm})(\alpha'_{\theta h}) - (\alpha_{\theta m})(\delta_{vh})}{(\alpha'_{\theta v})(\delta_{vh}) - (\delta_{vv})(\alpha'_{\theta h})}\right] M_a$$

Substituting values previously obtained for

$EI\delta'_{vm}$, $EI\alpha'_{\theta h}$, etc.

$$V_a = \left[\frac{(\widehat{L}\cdot\frac{S}{2})\left[\widehat{L}\cdot\left[Rise-(R-\bar{y})\right]\right] - (\widehat{L})\left[\widehat{L}\cdot\frac{S}{2}\cdot\left[Rise-(R-\bar{y})\right]\right]}{(\widehat{L}\cdot\frac{S}{2})\left[\widehat{L}\cdot\frac{S}{2}\left[Rise-(R-\bar{y})\right]\right] - \left[\int x^2 ds\right]\left[\widehat{L}-\left[Rise-(R-\bar{y})\right]\right]}\right] M_a$$

Since the numerator of the above expression is equal to zero, then $V_a = 0$

Determination of M_a as caused by spread ΔH

Substituting $V_a = 0$ in equation A

we obtain $H_a = (-)\ M_a\,\dfrac{(\delta'vm)}{(\delta vh)}$

Substituting $V_a = 0$ and $H_a = (-)\ M_a\,\dfrac{(\delta'vm)}{(\delta vh)}$

In equation (5), we obtain

$$EI\,\Delta H = EI\left[M_a(\delta'_{hm}) - M_a\left(\frac{\delta' vm}{\delta vh}\right)(\delta_{hh})\right]$$

$$M_a = \frac{EI\,\Delta H}{EI\delta'_{hm} - \dfrac{(\delta' vm)(\delta_{hh})(EI)^2}{EI\,\delta_{vh}}} = \frac{EI\,\Delta H}{D_4}$$

Determination of H_a as caused by spread ΔH

Substituting $V_a = 0$ and $M_a = (-)\frac{H_a(\delta_{vh})}{(\delta'_{vm})}$ in (5)

$$EI\,\Delta H = EI\ (-)\frac{H_a(\delta_{vh})}{(\delta'_{vm})}\ (\delta'_{hm}) + H_a(\delta_{hh})$$

$$H_a = \frac{EI\,\Delta H}{EI\,\delta_{hh} - \frac{(\delta_{vh})(\delta'_{hm})(EI)^2}{EI\,\delta'_{vm}}} = \frac{EI\,\Delta H}{D_5}$$

VIII(a) DETERMINATION OF THE RELATIONSHIP BETWEEN THE REACTIONS, INDUCED AS A RESULT OF SPREAD, AND THE SPAN FOR THE SAME RISE/SPAN RATIO.

Using the relationship between deflections and span that have been previously determined, and substituting in the appropriate equations, we obtain,

Relationship between V_a and Span for Arch 2

$$V_a = \frac{(EI)^2}{(EI)^2}\left[\frac{(\delta'_{vm})(\frac{S_2}{S_1})^2(\alpha'_{\theta h})(\frac{S_2}{S_1})^2 - (\alpha_{\theta m})(\frac{S_2}{S_1})(\delta_{vh})(\frac{S_2}{S_1})^3}{(\alpha'_{\theta v})(\frac{S_2}{S_1})^2(\delta_{vh})(\frac{S_2}{S_1})^3 - (\delta_{vv})(\frac{S_2}{S_1})^3(\alpha'_{\theta h})(\frac{S_2}{S_1})^2}\right] M_a$$

$$= \left[\frac{(\delta'_{vm})(\alpha'_{\theta h}) - (\alpha_{\theta m})(\delta_{vh})}{(\alpha'_{\theta v})(\delta_{vh}) - (\delta_{vv})(\alpha'_{\theta h})}\right]\frac{(\frac{S_2}{S_1})^4}{(\frac{S_2}{S_1})^5}\ M_a$$

Since the numerator of the term in the bracket has been found equal to zero, we find

$V_a = 0$ (regardless of Span)

Relationship between M_a and Span for Arch 2

$$M_a = \frac{EI\,\Delta H}{EI\,\delta'_{hm}(\frac{S_2}{S_1})^2 - \frac{(\delta'_{vm})(\frac{S_2}{S_1})^2(\delta_{hh})(\frac{S_2}{S_1})^3(EI)^2}{EI(\delta_{vh})(\frac{S_2}{S_1})^3}}$$

$$M_a = \frac{EI\,\Delta H}{\left[EI\delta'_{hm} - \frac{(\delta'_{vm})(\delta_{hh})(EI)^2}{EI\,(\delta_{vh})}\right](\frac{S_2}{S_1})^2}$$

$$= \frac{EI\,\Delta H}{D_4}(\frac{S_1}{S_2})^2$$

$\therefore$ M_a varies inversely as the square of the Span.

Relationship between H_a and Span for Arch 2

$$H_a = \frac{EI\,\Delta H}{EI\delta_{hh}(\frac{S_2}{S_1})^3 - \frac{(\delta_{vh})(\frac{S_2}{S_1})^3(\delta'_{hm})(\frac{S_2}{S_1})^2(EI)^2}{EI\delta'_{vm}(\frac{S_2}{S_1})^2}}$$

$$= \frac{EI\,\Delta H}{\left[EI\delta_{hh} - \frac{(\delta_{vh})(\delta'_{hm})(EI)^2}{EI\,\delta'_{vm}}\right](\frac{S_2}{S_1})^3}$$

$$= \frac{EI\,\Delta H}{D_5}\,(\frac{S_1}{S_2})^3$$

$\therefore$ H_a varies inversely as the cube of the Span.

DEVELOPMENT OF METHOD FOR NON-PRISMATIC ARCH

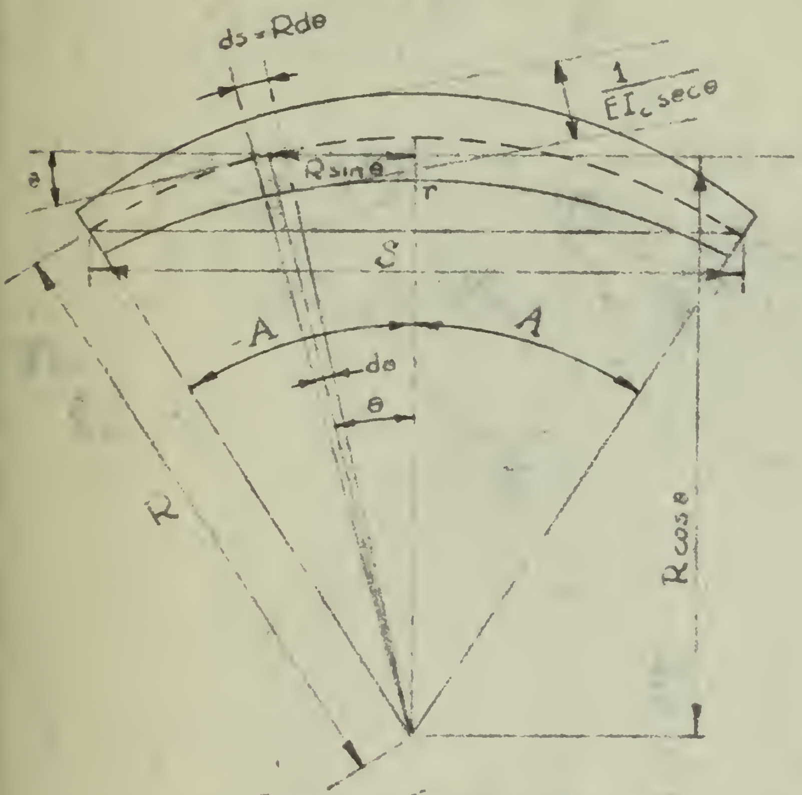
$$I_x = I_c \sec \theta$$

DERIVATIONS FOR FIXED END MOMENT, THRUST, AND SHEAR FOR A NON-PRISMATIC CIRCULAR ARCH, THE I AT ANY POINT BEING EQUAL TO THE I AT THE CROWN (I_c) TIMES THE SECANT OF THE ANGLE FORMED BY A TANGENT TO THE CENTERLINE OF THE ARCH AT THE POINT IN QUESTION AND A LINE PARALLEL TO THE SPRINGING LINE.

I. Radius in terms of Rise and Span

As derived for a Prismatic Arch: $R = \frac{4r^2 + S^2}{8r}$

II. Determination of the Centroid of the Non-Prismatic Arch Conjugate Structure (Plan View)



$$dA = \frac{ds}{EI} = \frac{Rd\theta}{EI_c \sec\theta}$$

$$\bar{x} = \frac{\int x\,da}{\int da}$$

$$\bar{x} = \frac{\int (R\sin\theta)\frac{Rd\theta}{EI_c \sec\theta}}{\int \frac{Rd\theta}{EI_c \sec\theta}} = \frac{R\int \sin\theta\cos\theta d\theta}{\int \cos\theta d\theta} = \frac{R\left[\frac{\sin^2\theta}{2}\right]_{-A}^{A}}{\left[\sin\theta\right]_{-A}^{+A}} = 0$$

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int R\cos\theta \frac{Rd\theta}{EI_c \sec\theta}}{\int \frac{Rd\theta}{EI_c \sec\theta}} = \frac{R\int \cos^2\theta d\theta}{\int \cos\theta d\theta}$$

$$= \frac{R\left[\frac{\theta + \sin\theta\cos\theta}{2}\right]_{-A}^{+A}}{\left[\sin\theta\right]_{A}^{A}} = \frac{\frac{R}{2}\left[\frac{L}{2R} + \frac{S}{2R}\left(\frac{R-r}{R}\right)\right] - \frac{R}{2}\left[-\frac{L}{2R} + \left(-\frac{S}{2R}\right)\left(\frac{R-r}{R}\right)\right]}{\left[\frac{S}{2R} - (-)\frac{S}{2R}\right]}$$

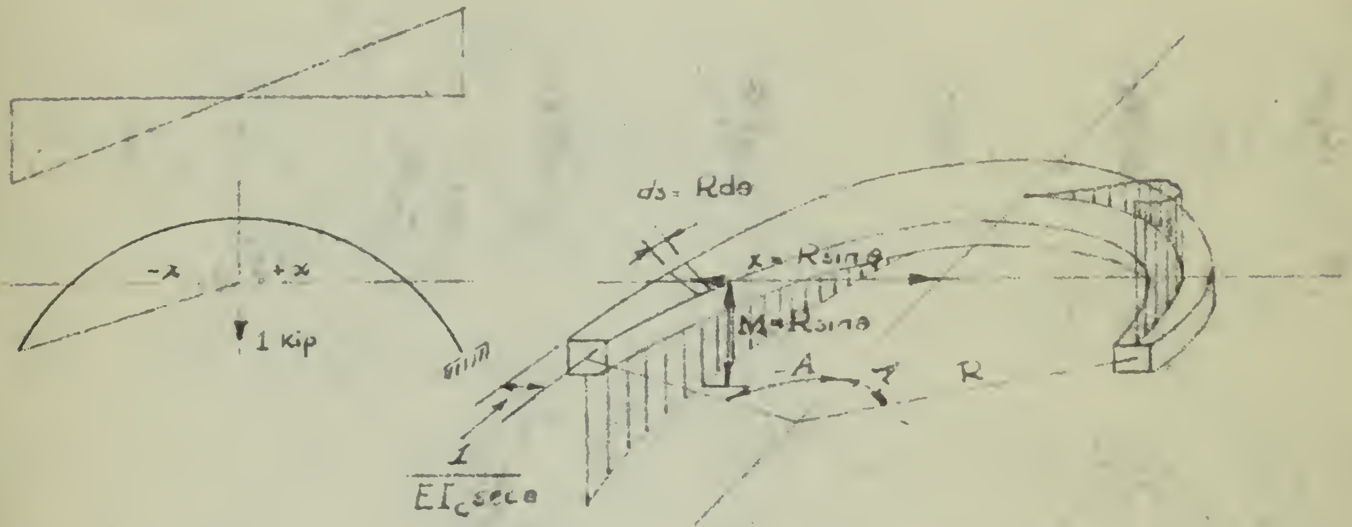
$$\bar{y} = \frac{R\left[\frac{\widehat{L}R + S(R-r)}{2R^2}\right]}{\frac{S}{R}} = \frac{\widehat{L}R + S(R-r)}{2S}$$

$\therefore$ For a symmetrical non-prismatic arch:

$$\bar{x} = 0 \qquad \bar{y} = \frac{\widehat{L}R + S(R-r)}{2S}$$

III. DETERMINATION OF DEFLECTION AT NEUTRAL POINT DUE TO:

(A) 1 kip vertical force at N.P.

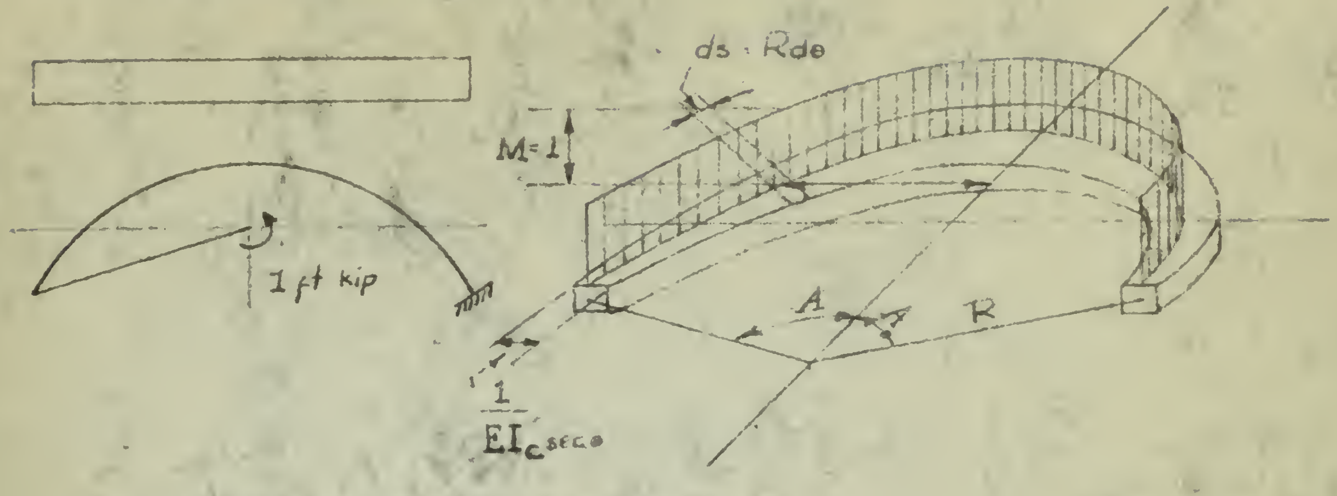


$$\delta_{vv} = \int_{-A}^{A} \frac{Mds}{EI}(x) =$$

$$EI_c\,\delta_{vv} = 2\int_0^A (R\sin\theta)\left(\frac{Rd\theta}{\sec\theta}\right)(R\sin\theta) = 2R^3\int_0^A \sin^2\theta\cos\theta d\theta$$

$$= 2R^3\left[\frac{\sin^3\theta}{3}\right]_0^A = \frac{2R^3}{3}\left[\frac{S}{2R}\right]^3 = \frac{2R^3S^3}{(3)(8)R^3} = \frac{S^3}{12}$$

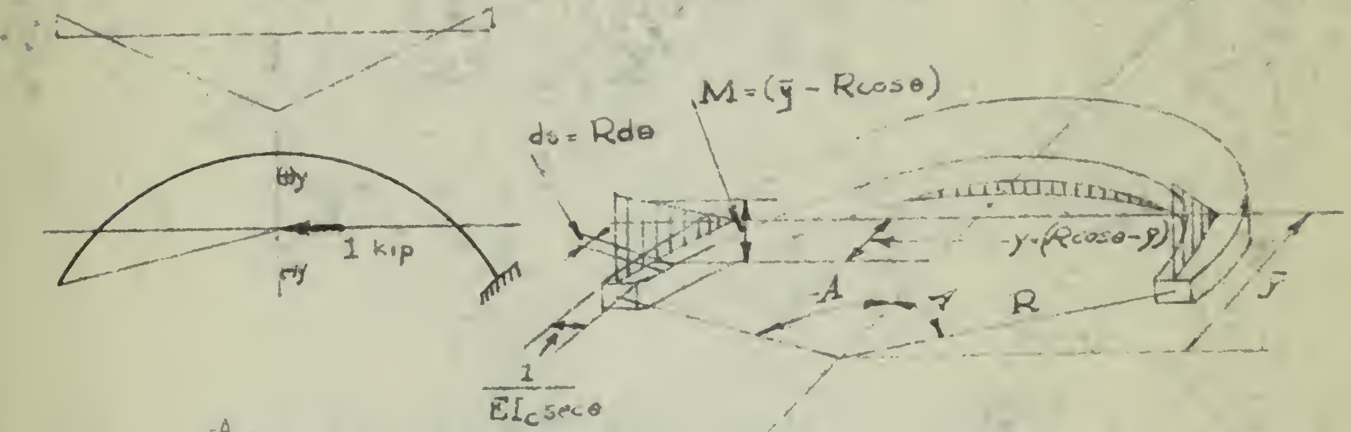
(B) 1 Ft.kip couple at N.P.



$$\alpha\theta m = \int_{-A}^{A} \frac{Mds}{EI}$$

$$EI_c \alpha\theta m = \int_{-A}^{A} \frac{M(Rd\theta)}{\sec\theta} = 2\int_{0}^{A} (1)\frac{Rd\theta}{\sec\theta} = 2\int_{0}^{A} R\cos\theta d\theta = 2R\left[\sin\theta\right]_{0}^{A} = 2R\left[\frac{S}{2R}\right] = S$$

(c) 1 kip horizontal force at N.P.



$$\delta hh = \int_{-A}^{A} \frac{Mds}{EI}(y)$$

$$EI_c \delta hh = 2\int_{0}^{A} (\bar{y}-R\cos\theta)\frac{Rd\theta}{\sec\theta}(R\cos\theta-\bar{y}) = 2\int_{0}^{A} (\bar{y}-R\cos\theta)R\cos\theta d\theta(R\cos\theta-\bar{y})$$

$$= 2\int_{0}^{A}\left[(-)(\bar{y})^2\cos\theta + 2(\bar{y})R\cos^2\theta - R^2\cos^3\theta\right] Rd\theta$$

$$= 2\left[(-)R(\bar{y})^2\int_{0}^{A}\cos\theta d\theta + 2R^2(\bar{y})\int_{0}^{A}\cos^2\theta d\theta - R^3\int_{0}^{A}\cos\theta d\theta + R^3\int_{0}^{A}\sin^2\cos\theta d\theta\right]$$

$$= 2\left[(-)R(\bar{y})^2\left[\sin\theta\right]_{0}^{A} + 2R^2\bar{y}\left[\left(\frac{\theta + \sin\theta\cos\theta}{2}\right)\right]_{0}^{A} - R^3\left[\sin\theta\right]_{0}^{A} + R^3\left[\frac{\sin^3\theta}{3}\right]_{0}^{A}\right]$$

$$= 2\left[\left[(-)R(\bar{y})^2\frac{S}{2R}\right] + \frac{2P^2(\bar{y})}{2}\left[\frac{\hat{L}}{2R} + \frac{S}{2R}\frac{(P-r)}{R}\right] - R^3\frac{S}{2R} + \frac{R^3}{3}\left(\frac{S}{2R}\right)^3\right]$$

$$= -(\bar{y})^2 S + (\bar{y})\left[\hat{L}R + S(R-r)\right] - R^2S + \frac{S^3}{12}$$

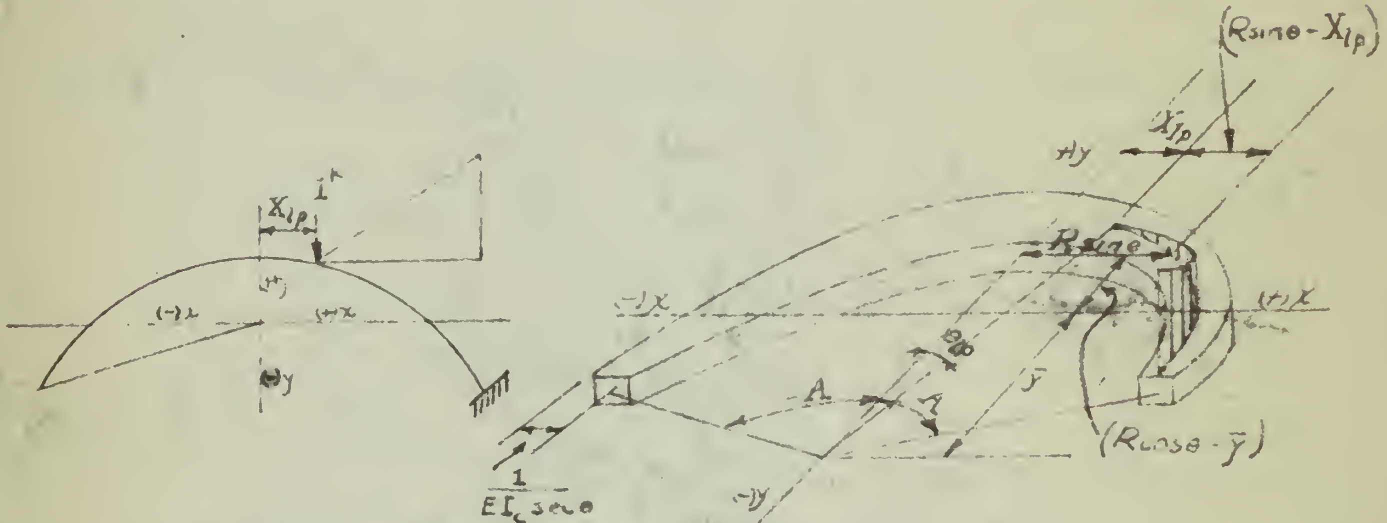
Since $\bar{y} = \frac{\hat{L}R + S(R-r)}{2S}$ we write the above as

$$= -(\bar{y})^2 S + (\bar{y})(\bar{y})(2S) - R^2S + \frac{S^3}{12}$$

$$= -(\bar{y})^2 S + 2(\bar{y})^2 S - R^2S + \frac{S^3}{12}$$

$$= (\bar{y})^2 S - R^2(S) + \frac{S^3}{12}$$

IV. DETERMINATION OF DEFLECTIONS AT THE NEUTRAL POINT (N.P.) CAUSED BY A 1 KIP VERTICAL LOAD ACTING ON THE ARCH AT A POINT THAT IS HORIZONTAL DISTANCE X_{1p} TO THE RIGHT OF THE CROWN.



(A) Determination of Vertical Deflection at N.P.

$$\Delta V = \int_{\theta_{1p}}^{A}\frac{Mds}{EI}(X)$$

$$EI_c\Delta V = \int_{\theta_{1p}}^{A}(R\sin\theta - X_{1p})\left(\frac{Rd\theta}{\sec\theta}\right)(R\sin\theta) = \int_{\theta}^{A}(R\sin\theta - X_{1p})R\cos\theta d\theta(R\sin\theta)$$

$$= \int_{\theta_{1p}}^{A}R^3\sin^2\theta\cos\theta d\theta - \int_{\theta_{1p}}^{A}R^2(X_{1p})\sin\theta\cos\theta d\theta$$

$$= R^3\left[\frac{\sin^3\theta}{3}\right]_{\theta_{1p}}^{A} - R^2(X_{1p})\left[\frac{\sin^2\theta}{2}\right]_{\theta_{1p}}^{A}$$

$$= \frac{R^3}{3}\left(\frac{S}{2R}\right)^3 - \frac{R^3}{3}\left(\sin^3\theta_{1p}\right) - \frac{R^2}{2}(X_{1p})\left(\frac{S}{2R}\right)^2 + \frac{R^2(X_{1p})}{2}\left(\sin^2\theta_{1p}\right)$$

Since $\sin\theta_{1p} = \frac{X_{1p}}{R}$ we write the above as

$$= \frac{S^3}{24} - \frac{R^3}{3}\left(\frac{X_{1p}}{R}\right)^3 - \frac{(X_{1p})S^2}{8} + \frac{R^2(X_{1p})}{2}\left(\frac{X_{1p}}{R}\right)^2$$

$$= \frac{S^3}{24} - \frac{(X_{1p})^3}{3} - \frac{(X_{1p})S^2}{8} + \frac{(X_{1p})^3}{2}$$

$$= \frac{S^3}{24} + \frac{(X_{1p})^3}{6} - \frac{(X_{1p})S^2}{8}$$

(B) DETERMINATION OF HORIZONTAL DEFLECTION AT H.P.

$$\Delta_H = \int_{\theta_{1p}}^{A} \frac{M\,ds}{EI}(y)$$

$$EI_c\Delta_H = \int_{\theta_{1p}}^{A} (R\sin\theta - X_{1p})\,\frac{R\,d\theta}{\sec\theta}\,(R\cos\theta - \bar{y}) = \int_{\theta_{1p}}^{A} (R\sin\theta - X_{1p})R\cos\theta(R\cos\theta - \bar{y})\,d\theta$$

$$= \int_{\theta_{1p}}^{A} R^3\cos^2\theta\sin\theta\,d\theta - \int_{\theta_{1p}}^{A} R^2\bar{y}\sin\theta\cos\theta\,d\theta - \int_{\theta_{1p}}^{A} R^2(X_{1p})\cos^2\theta\,d\theta$$

$$+ \int_{\theta_{1p}}^{A} R(\bar{y})(X_{1p})\cos\theta\,d\theta$$

$$= (-)R^3\left[\frac{\cos^3\theta}{3}\right]_{\theta_{1p}}^{A} - R^2(\bar{y})\left[\frac{\sin^2\theta}{2}\right]_{\theta_{1p}}^{A} - R^2(X_{1p})\left[\left(\frac{\theta + \sin\theta\cos\theta}{2}\right)\right]_{\theta_{1p}}^{A}$$

$$+ R(\bar{y})(X_{1p})\left[\sin\theta\right]_{\theta_{1p}}^{A}$$

$$= (-)\frac{R^3}{3}\left[\left(\frac{R-r}{R}\right)^3 - \cos^3\theta_{1p}\right] - \frac{R^2(\bar{y})}{2}\left[\left(\frac{S}{2R}\right)^2 - \sin^2\theta_{1p}\right] - \frac{R^2X_{1p}}{2}\left[\frac{L}{2R} + \frac{S}{2R}\left(\frac{R-r}{R}\right)\right]$$

$$+ \frac{R^2(X_{1p})}{2}\left[\theta_{1p} + \sin\theta_{1p}\cos\theta_{1p}\right] + R(\bar{y})(X_{1p})\left[\left(\frac{S}{2R}\right) - \sin\theta_{1p}\right]$$

$$= -\frac{(R-r)^3}{3} + \frac{R^3}{3}\cos^3\theta_{1p} - \frac{(\bar{y})S^2}{8} + \frac{R^2(\bar{y})}{2}\left(\frac{X_{1p}}{R}\right)^2 - \frac{(X_{1p})\left[LR + S(R-r)\right]}{4}$$

$$+\frac{R^2(X_{1p})}{2}\left[\theta_{1p}+\sin\theta_{1p}\cos\theta_{1p}\right]+\frac{\bar{y}(X_{1p})S}{2}-R(\bar{y})(X_{1p})\left(\frac{X_{1p}}{R}\right)$$

$$EI_c\Delta_H=\frac{(R-r)^3}{3}-\frac{(\bar{y})S^2}{8}+\frac{(\bar{y})(X_{1p})^2}{2}-(X_{1p})\left[\frac{\widehat{L}R+S(R-r)}{4}\right]+\frac{(\bar{y})(X_{1p})S}{2}-(\bar{y})(X_{1p})^2$$

$$+\frac{R^3}{3}\cos^3\theta_{1p}+\frac{R^2(X_{1p})}{2}\left[\theta_{1p}+\sin\theta_{1p}\cos\theta_{1p}\right]$$

Since $\bar{y}=\frac{\widehat{L}R+S(R-r)}{2S}$ we write the above as

$$\frac{(R-r)^3}{3}-\frac{(\bar{y})S^2}{8}+\frac{(\bar{y})(X_{1p})^2}{2}-\frac{(X_{1p})(\bar{y})(2S)}{4}+\frac{(\bar{y})(X_{1p})S}{2}-(\bar{y})(X_{1p})^2$$

$$+\frac{R^3}{3}\cos^3\theta_{1p}+\frac{R^2(X_{1p})}{2}\left[\theta_{1p}+\sin\theta_{1p}\cos\theta_{1p}\right]$$

$$=\frac{(R-r)^3}{3}-\bar{y}\left[\frac{S^2}{8}+\frac{(X_{1p})^2}{2}\right]+\frac{R^3}{3}\cos^3\theta_{1p}+\frac{R^2(X_{1p})}{2}\left[\theta_{1p}+\sin\theta_{1p}\cos\theta_{1p}\right]$$

(C) DETERMINATION OF ANGULAR DEFLECTION AT L.P.

$$\theta=\int_{\theta_{1p}}^{A}\frac{Mds}{EI}$$

$$EI_c\theta=\int_{\theta_{1p}}^{A}(R\sin\theta-X_{1p})\frac{Rd\theta}{\sec\theta}=\int_{\theta_{1p}}^{A}(R\sin\theta-X_{1p})R\cos\theta d\theta$$

$$=\int_{\theta_{1p}}^{A}R^2\sin\theta\cos\theta d\theta-\int_{\theta_{1p}}^{A}R(X_{1p})\cos\theta d\theta$$

$$=\frac{R^2}{2}\left[\sin^2\theta\right]_{\theta_{1p}}^{A}-R(X_{1p})\left[\sin\theta\right]_{\theta_{1p}}^{A}$$

$$=\frac{R^2}{2}\left[\left(\frac{S}{2R}\right)^2-\sin^2\theta_{1p}\right]-R(X_{1p})\left[\frac{S}{2R}-\sin\theta_{1p}\right]$$

$$=\frac{S^2}{8}-\frac{R^2}{2}\left(\frac{X_{1p}}{R}\right)^2-\frac{(X_{1p})S}{2}+R(X_{1p})\left(\frac{X_{1p}}{R}\right)$$

$$=\frac{S^2}{8}-\frac{(X_{1p})^2}{2}-\frac{(X_{1p})S}{2}+(X_{1p})^2$$

$$=\frac{S^2}{8}+\frac{(X_{1p})^2}{2}-\frac{(X_{1p})S}{2}$$

(D) THE DETERMINATION OF THE REACTIONS AT E.P.

$$V_o = (-)\frac{\Delta V}{\delta vv} \qquad H_o = (-)\frac{\Delta H}{\delta hh} \qquad M_o = (-)\frac{\theta}{\alpha\,\theta m}$$

$$V_o = (-)\,\frac{\frac{S^3}{24} + \frac{(X_{1p})^3}{6} - \frac{(X_{1p})S^2}{8}}{\frac{S^3}{12}}$$

$$H_o = \frac{\frac{(-)(R-r)^3}{3} - \bar{y}\left[\frac{S^2}{8} + \frac{(X_{1p})^2}{2}\right] + \frac{R^3}{3}\cos^3\theta_{1p} + \frac{R^2(X_{1p})}{2}\left[\theta_{1p} + \sin\theta_{1p}\cos\theta_{1p}\right]}{(\bar{y})^2 S - R^2(S) + \frac{S^3}{12}}$$

$$M_o = (-)\,\frac{\frac{S^2}{8} + \frac{(X_{1p})^2}{2} - \frac{(X_{1p})S}{2}}{S}$$

(E) THE DETERMINATION OF THE FIXED END MOMENTS

$$M_l^f = V_o\frac{S}{2} - H_o\left[r-(R-\bar{y})\right] + M_o$$

Note: $\bar{y} = \frac{\widehat{L}R + S(R-r)}{2S}$

M_r^f is determined by writing moments about the right end.

A positive value of moment indicates tension on the top of the arch.

V. DETERMINATION OF RELATIONSHIP BETWEEN MOMENT AND SPAN FOR THE SAME RISE/SPAN RATIO.

Arches of the same Rise/Span ratio have been proved similar on Page 17. The moment arms of corresponding $\frac{Mds}{EI}$ loads have been proved to vary as $\frac{S_2}{S_1}$. These relationships hold for the non-prismatic arch as well as for the prismatic since the working line of each is an arc of a circle.

From Page 14 it is seen that M at any point due to a 1 kip load on the Arch = $(R\sin\theta - M_{1p})$.

For Arch 2, $M_2 = R_2\sin\theta - R_2\sin\theta_{1p} = R_2(\sin\theta - \sin\theta_{1p})$

For Arch 1, $M_1 = R_1\sin\theta - R_1\sin\theta_{1p} = R_1(\sin\theta - \sin\theta_{1p})$

$$\frac{M_2}{M_1} = \frac{R_2}{R_1} = \frac{S_2}{S_1} \quad \text{or} \quad M_2 = M_1\left(\frac{S_2}{S_1}\right)$$

Note: The above M's are moment intensities acting on areas $\frac{ds}{EI_c\sec\theta}$, or the moment existing at the corresponding section of the cut back structure.

For Arch 1

$$\left[\frac{ds}{EI_c\sec\theta}\right]_1$$

For Arch 2

$$\left[\frac{ds}{EI_c\sec\theta}\right]_1 \times \frac{R_2}{R_1} = \left[\frac{ds}{EI_c\sec\theta}\right]_1 \times \frac{S_2}{S_1}$$

Since the moment intensities vary as $\frac{S_2}{S_1}$, and the areas on which these intensities act vary as $\frac{S_2}{S_1}$, the Elastic Loads $\left(\frac{Mds}{EI_c\sec\theta}\right)$ vary as $\left(\frac{S_2}{S_1}\right)^2$. The vertical and horizontal deflections at the N.P. caused by a load at any point on the arch consequently will vary as $\left(\frac{S_2}{S_1}\right)^3$.

Since the vertical and horizontal reactions at the N.P. of any arch are equal to the quotient of linear deflections, then the V_o and H_o will be independent of span. For a more rigid proof of this, see Page 18 .

For Arch 2

$$M_{o2} = -\frac{\theta_2}{\alpha_2} = \frac{\int_{\theta_{1p}}^{A}\left[\frac{Mds}{EI_c\sec\theta}\right]_2}{S_2} = \frac{\int_{\theta_{1p}}^{A}\left[\frac{Mds}{EI_c\sec\theta}\right]_1 \times \left(\frac{S_2}{S_1}\right)^2}{S_1 \times \frac{S_2}{S_1}}$$

For Arch 1

$$M_{o1} = -\frac{\theta_1}{\alpha_1} = \frac{\int_{\theta_{1p}}^{A}\left[\frac{Mds}{EI_c\sec\theta}\right]_1}{S_1}$$

$$\frac{M_{o2}}{M_{o1}} = \left(\frac{S_2}{S_1}\right)$$

$$M_{o2} = M_{o1}\frac{S_2}{S_1} \qquad \text{(Mo's vary directly as span)}$$

Having proved that the vertical and horizontal reactions at the N.P. are independent of span, and that the moment reaction is directly proportional to span, we find by taking moments at the left end of the arch that the fixed end moment varies directly as the span. This proof is identical with that found on Page 19 .

VI. DETERMINATION OF CARRY-OVER FACTOR

Cutting the arch at the left springing and applying redundants M_a, V_a and H_a at that point, the deflections at the left end will be as follows:

(1) $\Delta V = M_a \delta'_{vm} + V_a \delta_{vv} + H_a \delta_{vh}$

(2) $\Delta H = M_a \delta'_{hm} + V_a \delta_{hv} + H_a \delta_{hh}$

(3) $\theta = M_a \alpha_{\theta m} + V_a \alpha'_{\theta v} + H_a \alpha'_{\theta h}$

By multiplying both sides of equations (1), (2) and (3) by EI_c, these equations take the form:

(4) $EI_c \Delta V = M_a(EI_c \delta'_{vm}) + V_a(EI_c \delta_{vv}) + H_a(EI_c \delta_{vh})$

(5) $EI_c \Delta H = M_a(EI_c \delta'_{hm}) + V_a(EI_c \delta_{hv}) + H_a(EI_c \delta_{hh})$

(6) $EI_c \theta = M_a(EI_c \alpha_{\theta m}) + V_a(EI_c \alpha'_{\theta v}) + H_a(EI_c \alpha'_{\theta h})$

From the theories of virtual work and Maxwell's Law of Reciprocal Deflections:

(Refer to Page 21 for x and y values)

$$EI\,\delta'_{vm} = EI\,\alpha'_{\theta v} = \int_{-A}^{A} \frac{x\,ds}{\sec\theta} = \int_{-A}^{A} \left(\frac{S}{2} + R\sin\theta\right)\cos\theta\,R\,d\theta$$

$$= \frac{SR}{2}\int_{-A}^{A} \cos\theta\,d\theta + R^2\int_{-A}^{A} \sin\theta\cos\theta\,d\theta$$

$$= \frac{SR}{2}\Big[\sin\theta\Big]_{-A}^{A} + \frac{R^2}{2}\Big[\sin^2\theta\Big]_{-A}^{A}$$

$$= \frac{SR}{2}\left[\frac{S}{2R} - (-)\frac{S}{2R}\right] + \frac{R^2}{2}\Big[0\Big]$$

$$= \frac{SR}{2}\left(\frac{S}{R}\right) = \frac{S}{2}(S)$$

$$EI_c\delta'_{hm} = EI\alpha'_{\theta h} = \int_{-A}^{A}\frac{y\,ds}{\sec\theta} = \int_{-A}^{A}\left[R\cos\theta-(R-r)\right]\cos\theta R\,d\theta$$

$$= 2\int_{0}^{A}R^2\cos^2\theta\,d\theta - 2(R-r)R\int_{0}^{A}\cos\theta\,d\theta$$

$$= 2R^2\left[\frac{\theta+\sin\theta\cos\theta}{2}\right]_{0}^{A} - 2(R-r)R\left[\sin\theta\right]_{0}^{A}$$

$$= R^2\left[\frac{\widehat{L}}{2R}+\frac{S}{2R}\frac{(R-r)}{R}\right] - 2(R-r)R\left(\frac{S}{2R}\right)$$

$$= \frac{1}{2}\left[\widehat{L}R + S(R-r)\right] - S(R-r)$$

$$= \left[\frac{\widehat{L}R + S(R-r)}{2S} - (R-r)\right]S$$

$$EI_c\alpha_{\theta m} = \int_{-A}^{A}\frac{ds}{\sec\theta} = \int_{0}^{A}R\cos\theta\,d\theta = 2R\left[\sin\theta\right]_{0}^{A} = 2R\left(\frac{S}{2R}\right) = S$$

$$EI_{c}\delta_{va} = EI\delta_{hv} = \int_{-A}^{A}xy\frac{ds}{\sec\theta}$$

$$= \int_{-A}^{A}\left[\frac{S}{2}+R\sin\theta\right]\left[R\cos\theta-(R-r)\right]\cos\theta R\,d\theta$$

$$= \frac{SR^2}{2}\int_{-A}^{A}\cos^2\theta\,d\theta + R^3\int_{-A}^{A}\cos^2\theta\sin\theta\,d\theta - \frac{S}{2}(R-r)R\int_{-A}^{A}\cos\theta\,d\theta - R^2(R-r)\int_{-A}^{A}\sin\theta\cos\theta\,d\theta$$

$$= \frac{SR^2}{2}\left[\frac{\theta+\sin\theta\cos\theta}{2}\right]_{-A}^{A} - \frac{R^3}{3}\left[\cos^3\theta\right]_{-A}^{A} - \frac{S}{2}(R-r)R\left[\sin\theta\right]_{-A}^{A} - \frac{R^2(R-r)}{2}\left[\sin^2\theta\right]_{-A}^{A}$$

$$= \frac{S}{2}R^2\left[\frac{\widehat{L}}{2R}+\frac{S}{2R}\frac{(R-r)}{R}\right] - 0 - \frac{S}{2}(R-r)R\left(\frac{S}{R}\right) - 0$$

$$= \frac{S^2}{2}\left[\frac{\widehat{L}R + S(R-r)}{2S}\right] - \frac{S^2}{2}(R-r)$$

$$= \left(\frac{S}{2}\right)\left[\frac{\widehat{L}R + S(R-r)}{2S} - (R-r)\right](S)$$

$$EI_c\delta_{vv} = \int_{-A}^{A} x^2 \frac{ds}{\sec\theta} = \int_{-A}^{A} \left(\frac{S}{2} + R\sin\theta\right)^2 \cos\theta R\, d\theta$$

$$= \int_{-A}^{A} \left(\frac{S^2}{4} + SR\sin\theta + R^2\sin^2\theta\right)\cos\theta R\, d\theta$$

$$= \frac{S^2}{4} R \int_{-A}^{A} \cos\theta\, d\theta + SR^2 \int_{-A}^{A} \sin\theta\cos\theta\, d\theta + R^3 \int_{-A}^{A} \sin^2\theta\cos\theta\, d\theta$$

$$= \frac{S^2}{4}(R)\Big[\sin\theta\Big]_{-A}^{A} + \frac{S(R^2)}{2}\Big[\sin^2\theta\Big]_{-A}^{A} + R^3\left[\frac{\sin^3\theta}{3}\right]_{-A}^{A}$$

$$= \frac{S^2}{4}(R)\left(\frac{S}{R}\right) + \frac{S(R^2)}{2}\Big[0\Big] + \frac{R^3}{3}\left[\left(\frac{S}{2R}\right)^3 + \left(\frac{S}{2R}\right)^3\right]$$

$$= \frac{S^3}{4} + \frac{2R^3}{3} \frac{S^3}{8R^3}$$

$$= \frac{S^3}{4} + \frac{S^3}{12} = \frac{S^3}{3}$$

$$EI_c\delta_{hh} = \int_{-A}^{A} y^2 \frac{ds}{EI} = 2\int_{0}^{A} \Big[R\cos\theta - (R-r)\Big]^2 \cos\theta R\, d\theta$$

$$= 2\int_{0}^{A} R^3\cos^3\theta - 2\int_{0}^{A} 2R^2(R-r)\cos^2\theta\, d\theta + 2\int_{0}^{A} R(R-r)^2\cos\theta\, d\theta$$

$$= 2R^3\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^A - 4R^2(R-r)\left[\frac{\theta + \sin\theta\cos\theta}{2}\right]_0^A + 2R(R-r)^2\Big[\sin\theta\Big]_0^A$$

$$= 2R^3\left(\frac{S}{2R}\right) - \frac{2R^3}{3}\left[\frac{S^3}{8R^3}\right] - \frac{4R^2(R-r)}{2}\left[\frac{\widehat{L}}{2R} + \frac{S}{2R}\,\frac{(R-r)}{R}\right] + 2R(R-r)^2\left(\frac{S}{2R}\right)$$

$$= SR^2 - \frac{S^3}{12} - \frac{4R^2(R-r)}{2}\left[\frac{\widehat{L}R + S(R-r)}{2R^2}\right] + S(R-r)^2$$

$$= SR^2 - \frac{S^3}{12} - (R-r)\Big[\widehat{L}R + S(R-r)\Big] + S(R-r)^2$$

$$= SR^2 - \frac{S^3}{12} - \widehat{L}R(R-r)$$

To obtain the carry-over factor we hinge the arch at the left end, set equations (4) and (5) equal to zero and solve simultaneously for V_a and H_a in terms of M_a. Then by taking moments at the fixed end, M_b can be expressed as the carry-over factor times M_a.

The resulting equations for V_a and H_a

$$V_a = D_2 M_a$$

$$H_a = D_1 M_a$$

will be the same as those for the prismatic arch found on Page 22 . The numerical values of D_2 and D_1 will be different due to the difference in the values of the deflections. Hence for the non-prismatic arch, we write

$$V_a = D_2' M_a$$

$$H_a = D_1' M_a$$

taking moments at the fixed end

$$M_b = M_a + V_a S$$

$$= M_a + (D_2')M_a S = (1 + D_2'S)M_a$$

$\therefore$ The Carry-Over Factor = (-)(1 + $D_2'S$)

VI.(a) DETERMINATION OF RELATIONSHIP BETWEEN CARRY-OVER FACTOR AND SPAN FOR THE SAME RISE/SPAN RATIO.

Prismatic arches of any span length, having the same rise/span ratio have been proved to be similar. The non-prismatic arches of the same rise/span ratio are also similar since secθ is the same for the corresponding points on the arches. Since the similarity holds for

non-prismatic arches the deflections for non-prismatic arches will vary with span in the same ratio as do the deflections for the prismatic arch (See Page 24).

For the Non-Prismatic Arch

$$V_a = D_2'(\frac{S_1}{S_2})\ M_a$$

$$H_a = D_1'(\frac{S_1}{S_2})\ M_a$$

Taking moments at the fixed end of Arch 2 of Span $S_2 =$

$$S_1(\frac{S_2}{S_1})$$

$$M_b = M_a + V_aS_2 = M_a + D_2'(\frac{S_1}{S_2})M_aS_2$$

$$= M_a + D_2'(\frac{S_1}{S_2})\ M_a\ S_1\ (\frac{S_2}{S_1})$$

$$= M_a + D_2'(S_1)\ M_a$$

$$= (1 + D_2'S_1)\ M_a$$

The Carry-Over Factor for Arch 2 = (-)$(1 + D_2'S_1)$ which is the same as the C.O.F. for Arch.1.

∴ Carry-Over Factor for arches of same Rise/Span ratio is independent of span.

VII. DETERMINATION OF ABSOLUTE STIFFNESS

The equation for the absolute stiffness of the non-prismatic arch would take the same form as the equation on Page 25 for the prismatic arch. The numerical value of D_3 would be different and the I for the non-prismatic would

be expressed as the I at the crown, I_c.

Hence for the absolute stiffness of the non-prismatic arch

$$K_a = \frac{EI_c}{D_3'} = \text{Absolute Stiffness}$$

VII(a) DETERMINATION OF RELATIONSHIP BETWEEN ABSOLUTE STIFFNESS AND SPAN FOR THE SAME RISE/SPAN RATIO

Since the deflections for the non-prismatic arch vary with span in the same ratio as do the deflections for the prismatic arch, the equation for the absolute stiffness for arch 2 written in terms of the absolute stiffness for arch 1 is in form the same as the equation on Page 26 .

$$K_a = \frac{EI_c}{D_3'} \cdot \frac{S_1}{S_2}$$

∴ Absolute Stiffness varies inversely as the span.

VIII. DETERMINATION OF THE INDUCED MOMENT AND REACTIONS (VERTICAL AND HORIZONTAL) OCCURRING AT THE END OF AN ARCH DUE TO SPREADING OF THE ARCH UNDER LOAD.

The equation for V_a and H_a in terms of Δ_a, for the non-prismatic arch take the same form as those equations found on Page 22 .

By substituting the values of the deflections in the numerator of the equation for V_a we prove that $V_a = 0$ for the non-prismatic arch also.

$$\frac{S}{2}(S)\left[\frac{\widehat{L}H + S(R-r)}{2S} - (R-r)\right]S - S\left(\frac{S}{2}\right)\left[\frac{\widehat{L}H + S(R-r)}{2S} - (R-r)\right]S = 0$$

Since $V_a = 0$, the equations for M_a, and H_a in terms of spread Δ_H will take the same form as the equations on Pages 27-28.

Modified for the non-prismatic arch, the equations are:

$$M_a = \frac{EI_c \Delta_H}{D_4'}$$

$$H_a = \frac{EI_c \Delta_H}{D_5'}$$

VIII(a). DETERMINATION OF THE RELATIONSHIP BETWEEN THE REACTIONS, INDUCED AS A RESULT OF SPREAD, AND THE SPAN FOR THE SAME RISE/SPAN RATIO.

Since the deflections for the non-prismatic arch vary with span in the same ratio as do the deflections for the prismatic arch, the equations for M_a and H_a for Arch 2 written in terms of their values for Arch 1 are

$$M_a = \frac{EI_c \Delta_H}{D_4'} \left(\frac{S_1}{S_2}\right)^2$$

$$H_a = \frac{EI_c \Delta_H}{D_5'} \left(\frac{S_1}{S_2}\right)^3$$

$\therefore M_a$ varies inversely as the square of the span.

$\therefore H_a$ varies inversely as the cube of the span.

RESULTS

CONCLUSIONS

In this thesis, we have presented data in both curve and tabular form that will enable an engineer to solve any circular arched bent, the arched member of which

(a) has a rise/span ratio between 0.04 and 0.40,

(b) is of constant moment of inertia or,

(c) is of varying moment of inertia, provided the moment of inertia at any point is equal to the "I" at the crown times the secant of the angle formed by a tangent to the center line of the arch at the point in question and a line parallel to the springing line.

"Fixed-end-moment" influence lines have been plotted by using the moments as caused by concentrated loads. By taking a summation of the ordinates of the influence lines, coefficients have been determined for formulas expressing the fixed-end-moment as caused by a complete uniform load; therefore, the arches may be solved for either concentrated or uniform loads.

Should the designer desire the required factors to a degree of accuracy greater than is available from the curves, exact formulas that will yield the factors are provided. A

complete solution for all factors can be obtained by knowing r (the rise in ft.), S (the span in ft.), and X_{1p} (the horizontal distance from the crown to the point of application of each load.) All formulas are based on loading to the right of the crown.

As a result of this investigation, the following relationships were proved:

For the prismatic arches having the same rise/span ratio:

(1) The shear and thrust are constant for all spans.

(2) The F.E.M. is directly proportional to span.

(3) The carry-over factor is constant for all spans.

(4) The absolute stiffness varies inversely as the span.

For the non-prismatic arches having the same rise/span ratio:

(1) The shear and thrust are constant for all spans.

(2) The F.E.M. is directly proportional to span.

(3) The carry-over factor is constant for all spans.

(4) The absolute stiffness varies inversely as the span.

For the non-prismatic arches regardless of rise/span ratio:

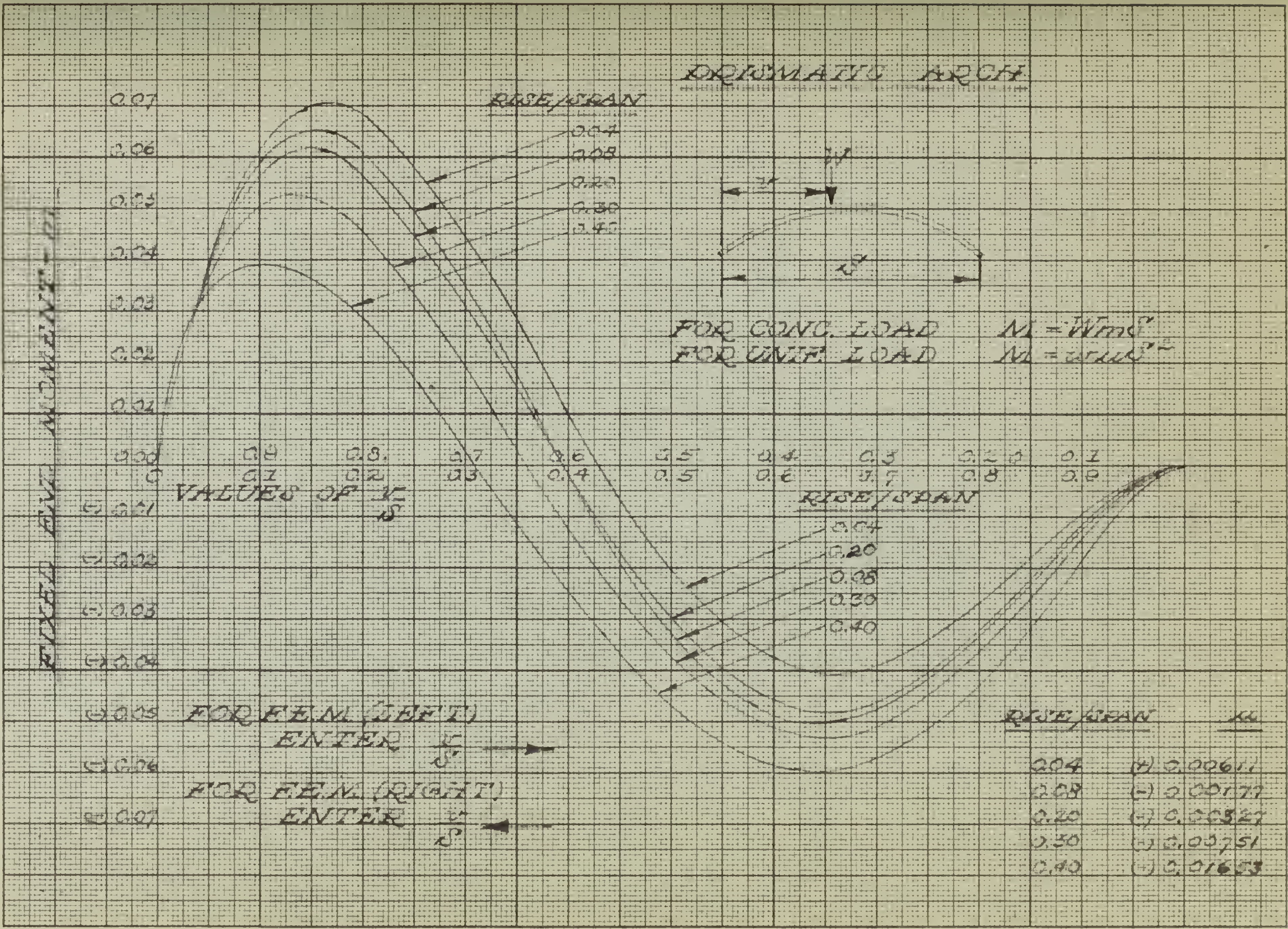
(1) The shear is constant.

From the two problems illustrating the application of the method, it is evident that with the required factors available, the engineer can solve a circular arched bent in practically the same time that he would need in solving a rectangular bent.

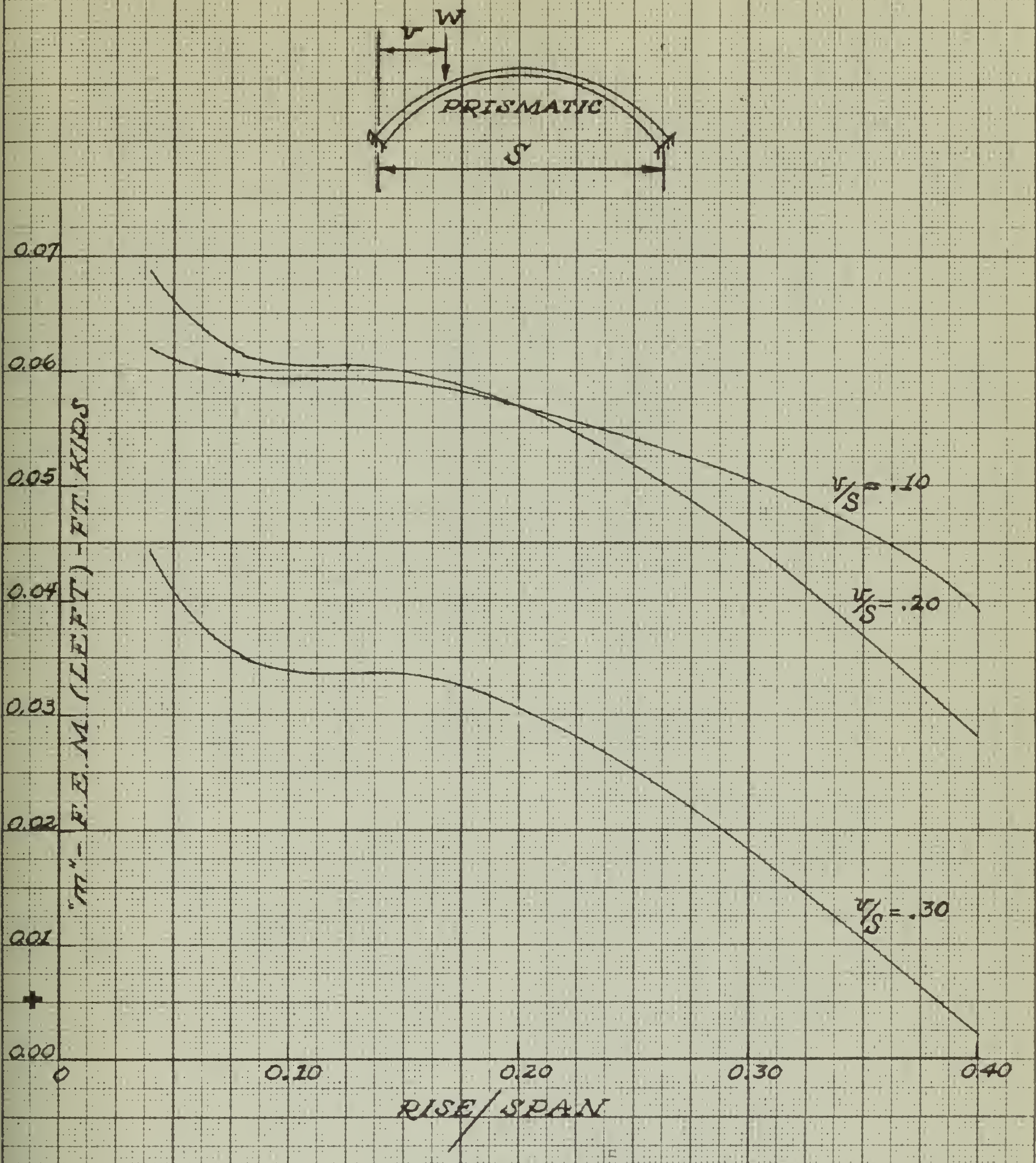
Although we have limited our application to a single

bent, the factors supplied in this thesis can be of particular advantage in extending the application of the moment distribution method to a series of continuous arches.

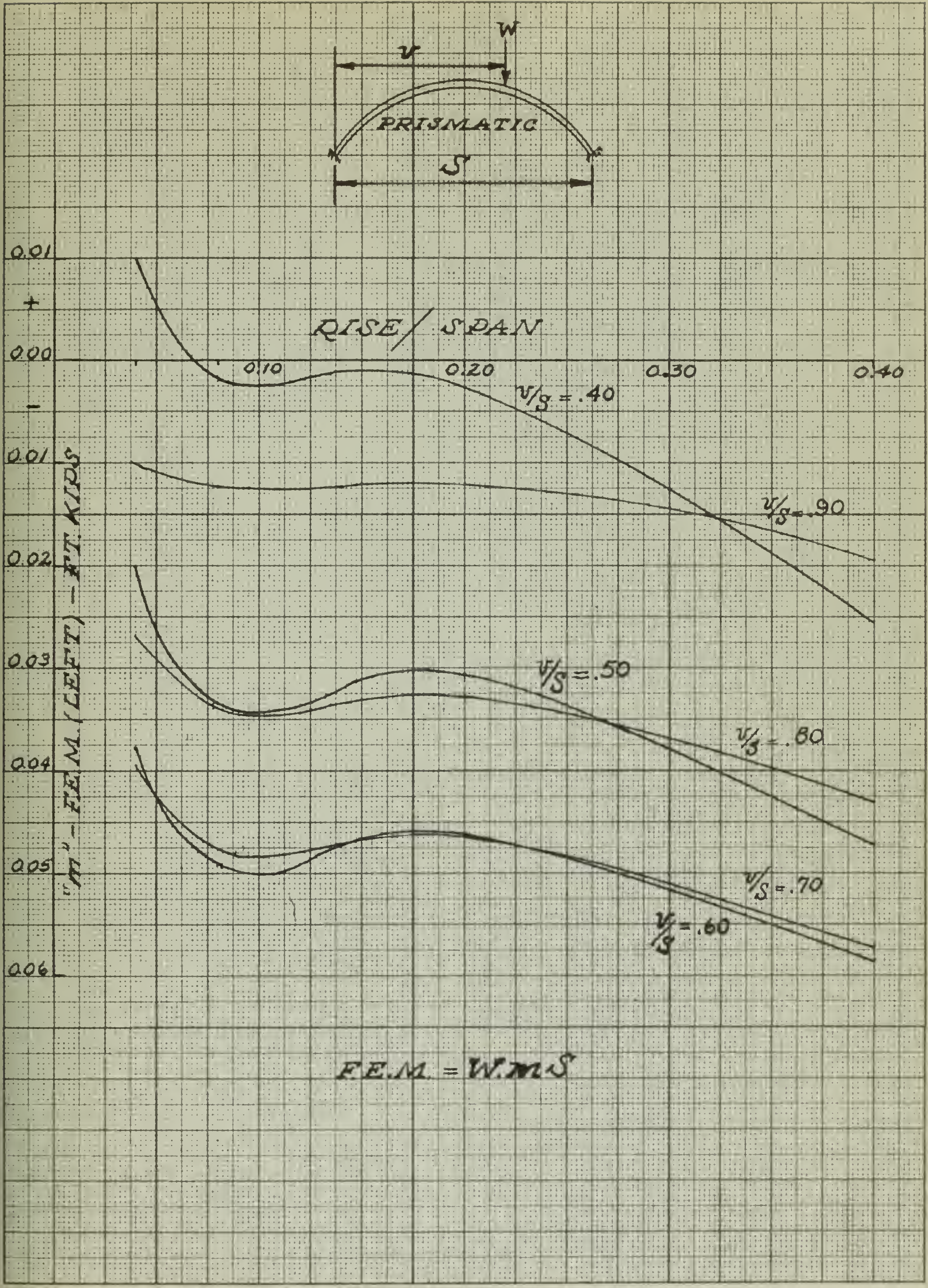
CURVES FOR PRISMATIC ARCH

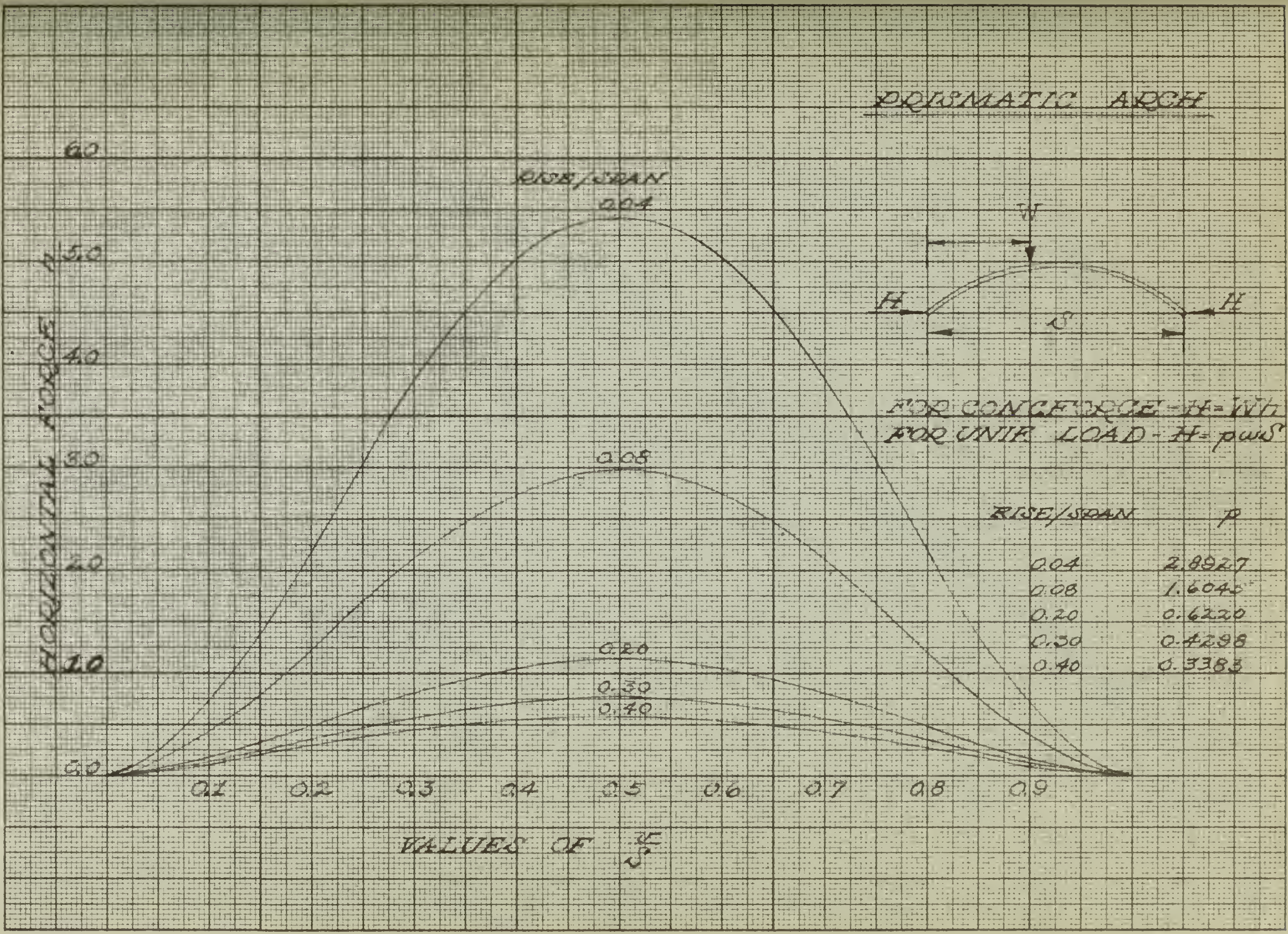


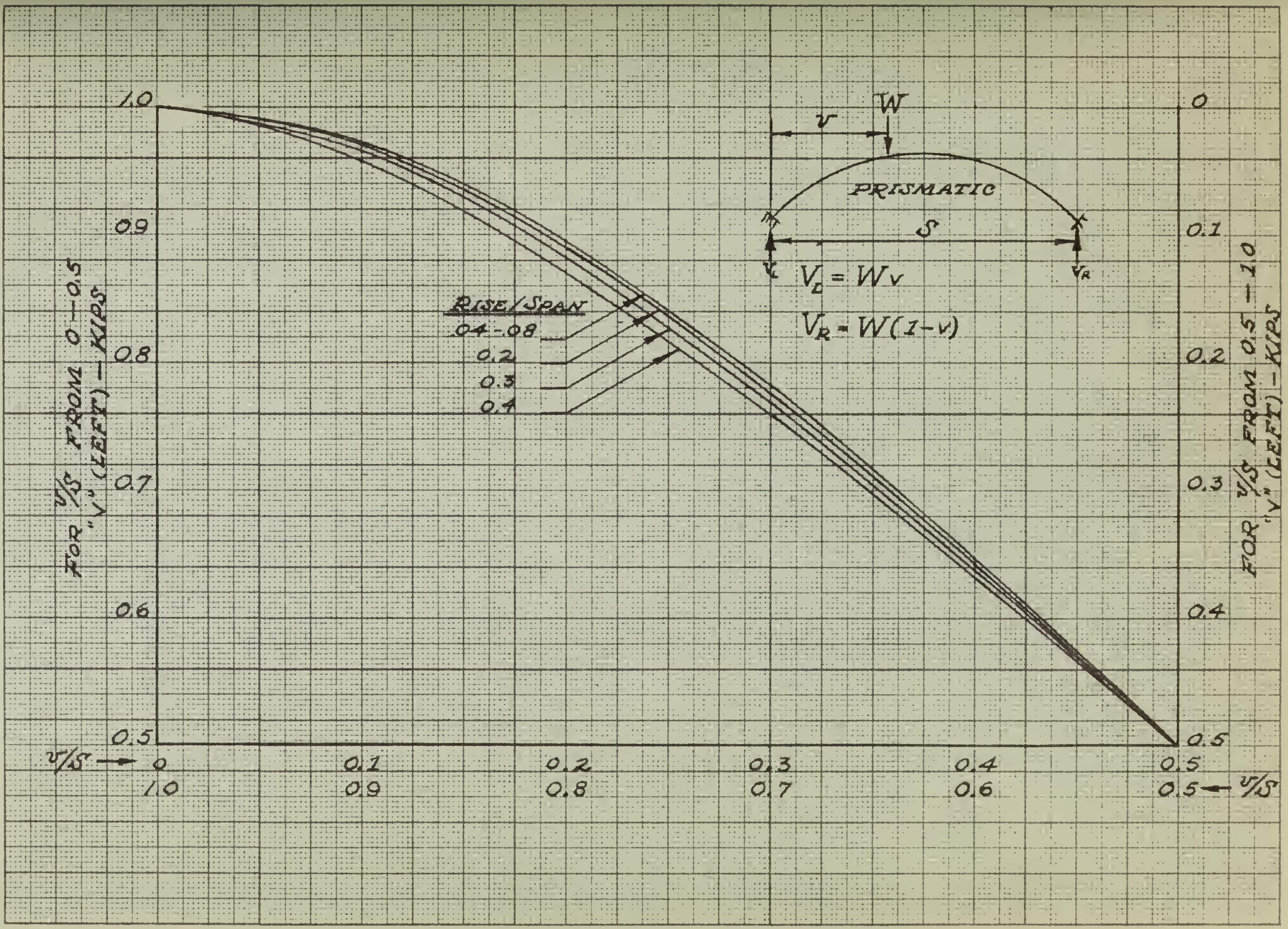
RISE/SPAN	m
0.04	(+) 0.00611
0.08	(-) 0.00177
0.20	(-) 0.00527
0.30	(-) 0.00751
0.40	(-) 0.01653

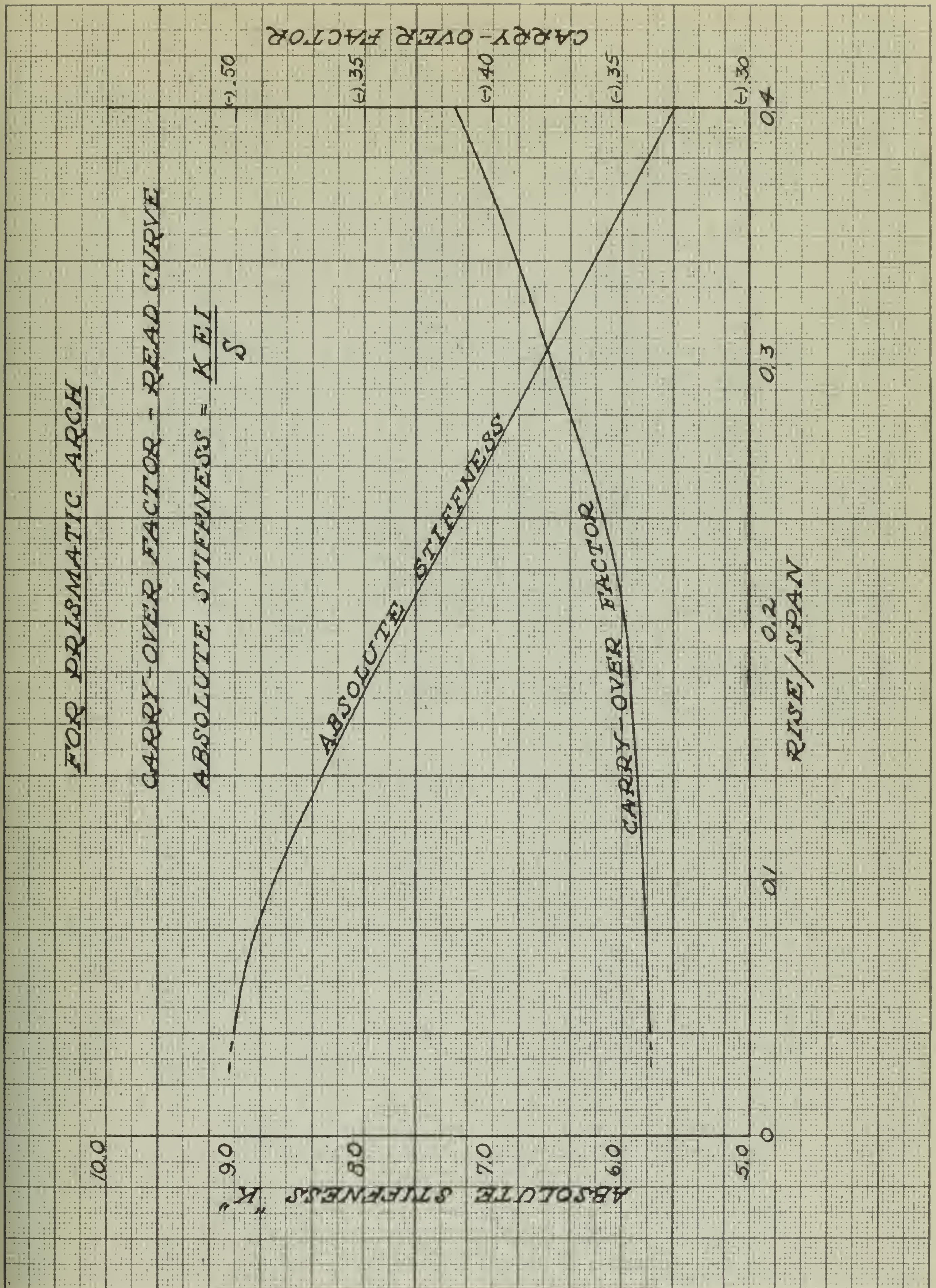

v
W
PRISMATIC
S
0.07
0.06
0.05
0.04
0.03
0.02
0.01
+
0.00
"m" - F.E.M. (LEFT) - FT. KIPS
0
0.10
0.20
0.30
0.40
RISE/SPAN
v/S = .10
v/S = .20
v/S = .30

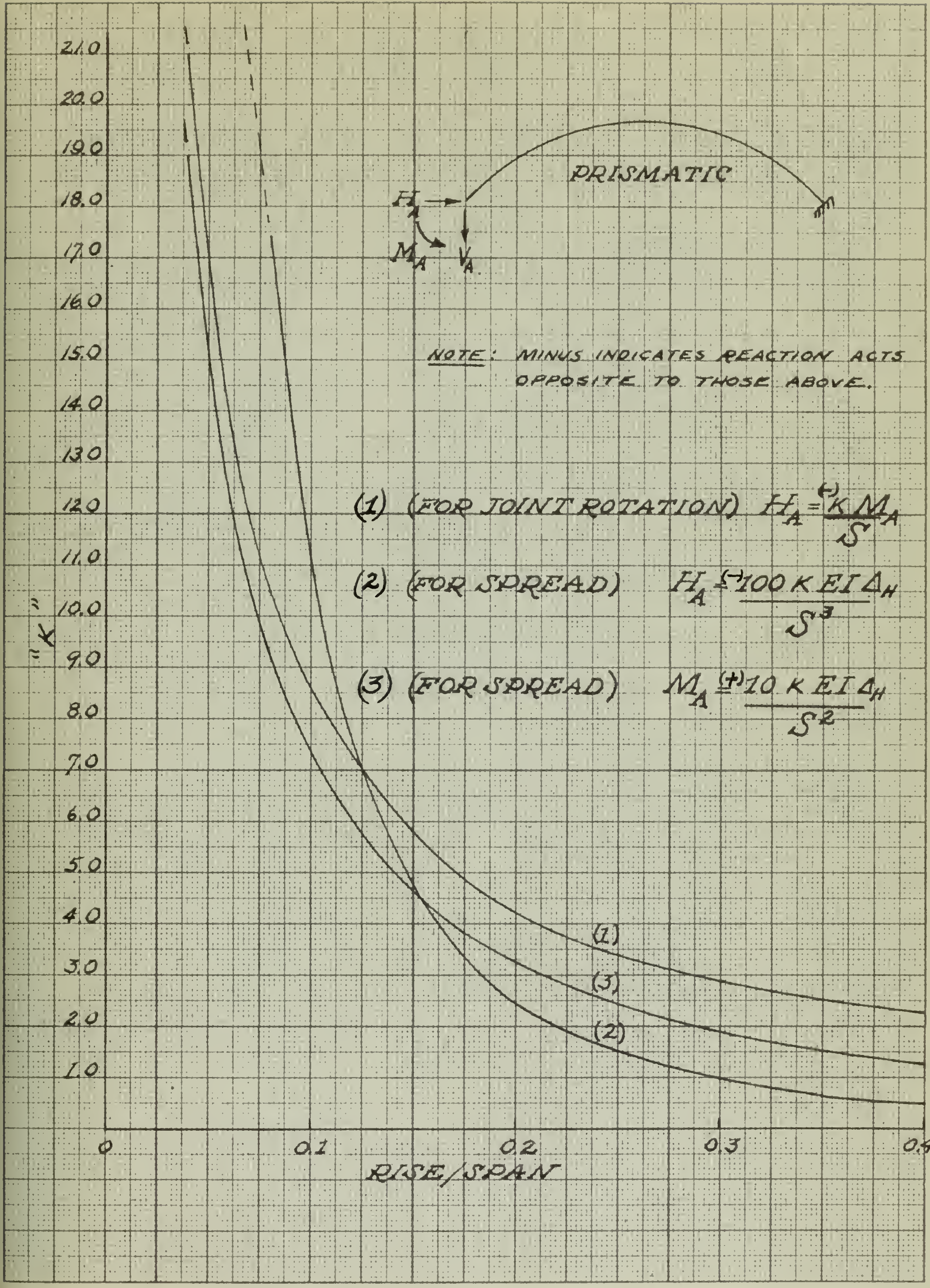
F.E.M. = WmS


W
v
PRISMATIC
S
0.01
+
0.00
-
0.01
0.02
0.03
0.04
0.05
0.06
RISE/SPAN
0.10
0.20
0.30
0.40
"m" - F.E.M. (LEFT) - FT. KIPS
v/S = .40
v/S = .90
v/S = .50
v/S = .80
v/S = .70
v/S = .60
F.E.M. = W.m S


PRISMATIC ARCH
W
H
H
S
FOR CONC. FORCE - H = Wh
FOR UNIF. LOAD - H = pwS
RISE/SPAN p
0.04 2.8927
0.08 1.6045
0.20 0.6220
0.30 0.4298
0.40 0.3383
RISE/SPAN
0.04
0.08
0.20
0.30
0.40
HORIZONTAL FORCE h
0.0
1.0
2.0
3.0
4.0
5.0
6.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
VALUES OF x/S


PRISMATIC
W
v
S
V_L
V_R
$V_L = Wv$
$V_R = W(1-v)$
RISE/SPAN
.04-.08
0.2
0.3
0.4
FOR v/S FROM 0 — 0.5 "V" (LEFT) — KIPS
FOR v/S FROM 0.5 — 1.0 "V" (LEFT) — KIPS
1.0
0.9
0.8
0.7
0.6
0.5
0
0.1
0.2
0.3
0.4
0.5
v/S →
0
0.1
0.2
0.3
0.4
0.5
1.0
0.9
0.8
0.7
0.6
0.5 ← v/S


FOR PRISMATIC ARCH
CARRY-OVER FACTOR - READ CURVE
ABSOLUTE STIFFNESS = $\frac{K\,EI}{S}$
ABSOLUTE STIFFNESS
CARRY-OVER FACTOR
CARRY-OVER FACTOR
(-).50
(-).35
(-).40
(-).35
(-).30
ABSOLUTE STIFFNESS "K"
10.0
9.0
8.0
7.0
6.0
5.0
RISE/SPAN
0
0.1
0.2
0.3
0.4


21.0
20.0
19.0
18.0
17.0
16.0
15.0
14.0
13.0
12.0
11.0
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
K
0
0.1
0.2
0.3
0.4
RISE/SPAN
PRISMATIC
H_A
M_A
V_A
NOTE: MINUS INDICATES REACTION ACTS OPPOSITE TO THOSE ABOVE.
(1) (FOR JOINT ROTATION) $H_A \overset{(-)}{=} \frac{K M_A}{S}$
(2) (FOR SPREAD) $H_A \overset{(-)}{=} \frac{100 K EI \Delta_H}{S^3}$
(3) (FOR SPREAD) $M_A \overset{(+)}{=} \frac{10 K EI \Delta_H}{S^2}$
(1)
(3)
(2)

RESULTS FOR PRISMATIC ARCH
BASED ON UNIT SPAN

TABULATION OF CONSTANTS FOR A PRISMATIC MEMBER BASED ON A UNIT SPAN

Rise/Span = 0.04

Load Point	Fem (Left) Ft.KIPS	Fem (Right) Ft.KIPS	V (Left)	H (Thrust)
1	+.061961 S	-.009910 S	.971391	.710188
2	+.068696 S	-.026942 S	.895637	2.214017
3	+.044320 S	-.039502 S	.783822	3.841791
4	+.009369 S	-.037813 S	.647783	5.015055
5	-.020077 S	-.020077 S	.500000	5.438771
Rise/Span = 0.08				
1	+.059115 S	-.012327 S	.971142	.403064
2	+.061345 S	-.035929 S	.895274	1.238693
3	+.034976 S	-.047651 S	.782626	2.077085
4	-.002109 S	-.049353 S	.647244	2.718642
5	-.033471 S	-.033471 S	.500000	2.956311
Rise/Span = 0.20				
1	+.056678 S	-.012916 S	.968894	.165382
2	+.056884 S	-.032323 S	.888650	.493918
3	+.030539 S	-.046434 S	.777114	.820233
4	-.002575 S	-.046224 S	.643649	1.048354
5	-.030651 S	-.030651 S	.500000	1.130556
Rise/Span = 0.30				
1	+.050550 S	-.014421 S	.965001	.125838
2	+.045027 S	-.036985 S	.882010	.353311
3	+.018267 S	-.051105 S	.769372	.567816
4	-.012708 S	-.051624 S	.638916	.711606
5	-.037928 S	-.037928 S	.500000	.761935
Rise/Span = 0.40				
1	+.035990 S	-.019356 S	.956347	.117985
2	+.028206 S	-.043029 S	.871235	.293130
3	+.002170 S	-.057118 S	.753339	.444331
4	-.025480 S	-.058521 S	.633042	.543263
5	-.047195 S	-.047193 S	.500000	.577442

Tabulation of Results for Prismatic Arch

Rise/Span	Absolute Stiffness	Carry-over Factor
0.04	9.009735 EI/S	(-) .3391362
0.08	8.834829 EI/S	(-) .3406277
0.20	7.695889 EI/S	(-) .3488530
0.30	6.631018 EI/S	(-) .3777210
0.40	5.591090 EI/S	(-) .4148421

With Joint A free to rotate, and a moment of M_a ft.kips applied at A, the following thrusts and shears are induced at A.

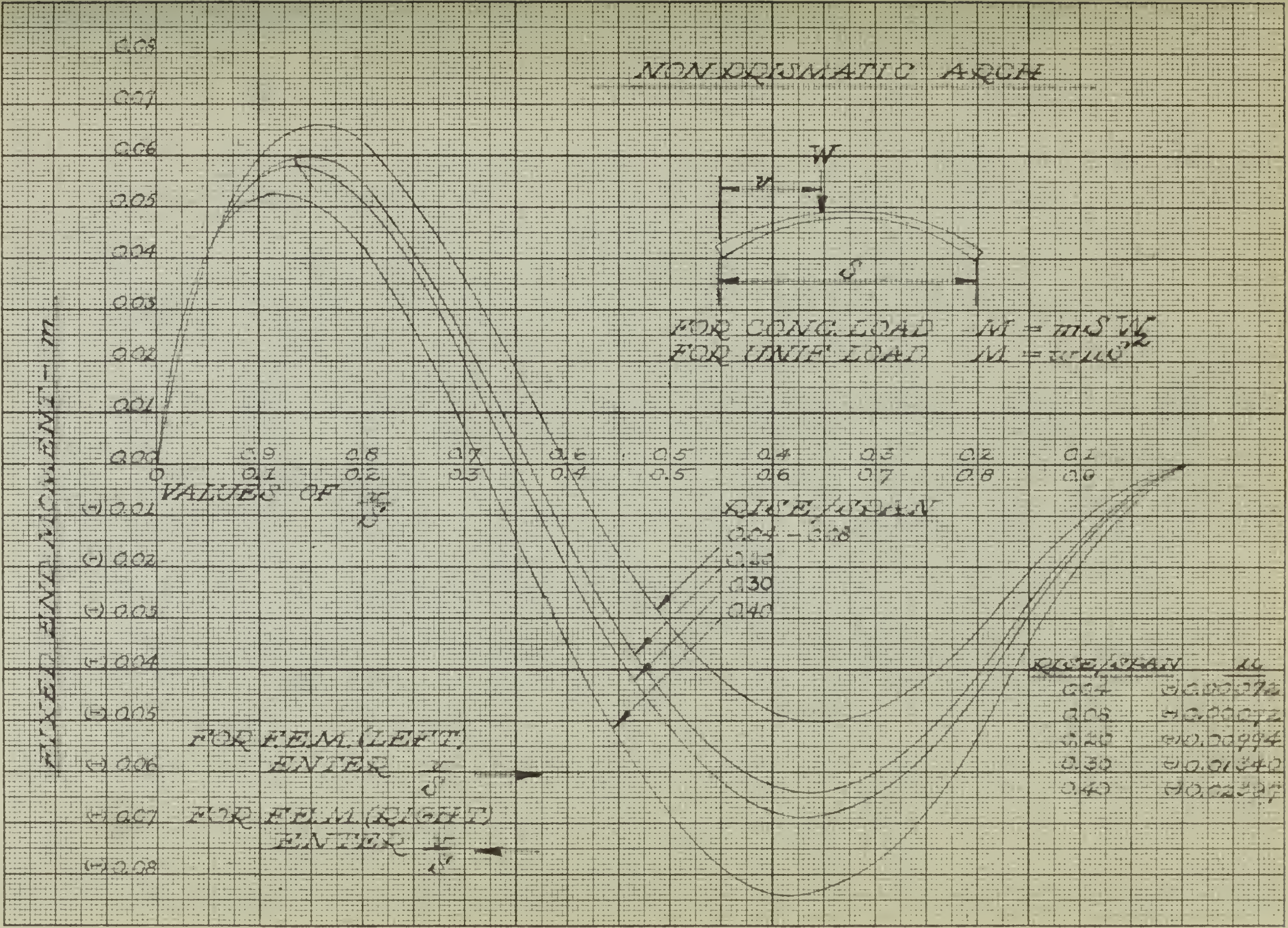
Rise/Span	Thrust (KIPS)	Vertical Shear (KIPS)
0.04	20.937421 M_a/S	.6606638 M_a/S
0.08	10.408726 M_a/S	.6583723 M_a/S
0.20	4.216799 M_a/S	.6511470 M_a/S
0.30	2.889591 M_a/S	.6222790 M_a/S
0.40	2.240648 M_a/S	.5851579 M_a/S

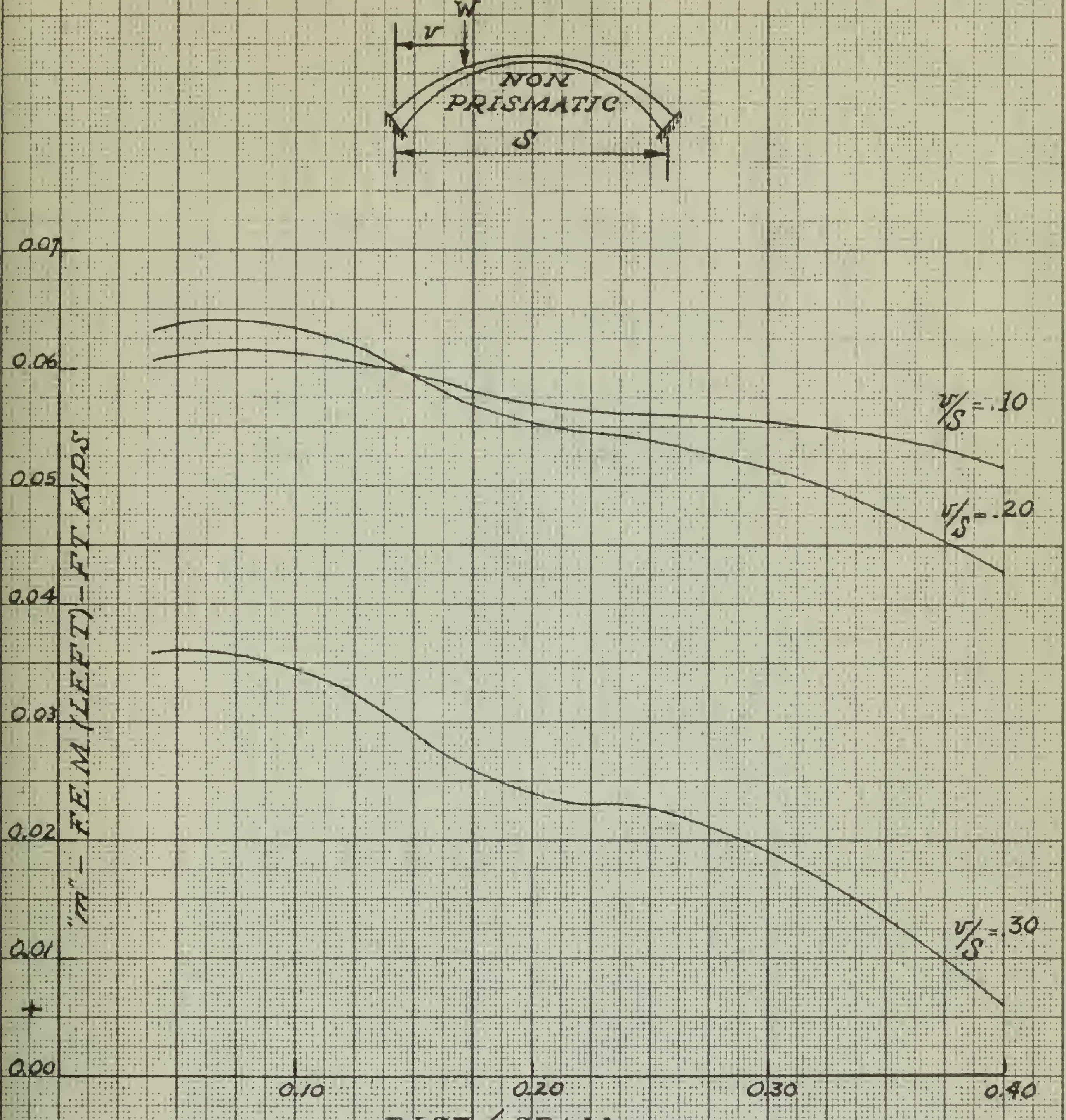
When a spread of ΔH feet occurs, the following moments, shears and thrusts occur.

Rise/Span	Thrust (KIPS)	Moment (Ft.KIPS)	Shear
0.04	7064.9449 $EI\,\Delta H/S^3$	(-)188.64038 $EI\,\Delta H/S^2$	0
0.08	1740.5620 $EI\,\Delta H/S^3$	(-) 92.67798 $EI\,\Delta H/S^2$	0
0.20	245.8247 $EI\,\Delta H/S^3$	(-) 32.45204 $EI\,\Delta H/S^2$	0
0.30	97.8593 $EI\,\Delta H/S^3$	(-) 19.16094 $EI\,\Delta H/S^2$	0
0.40	48.5544 $EI\,\Delta H/S^3$	(-) 12.52766 $EI\,\Delta H/S^2$	0

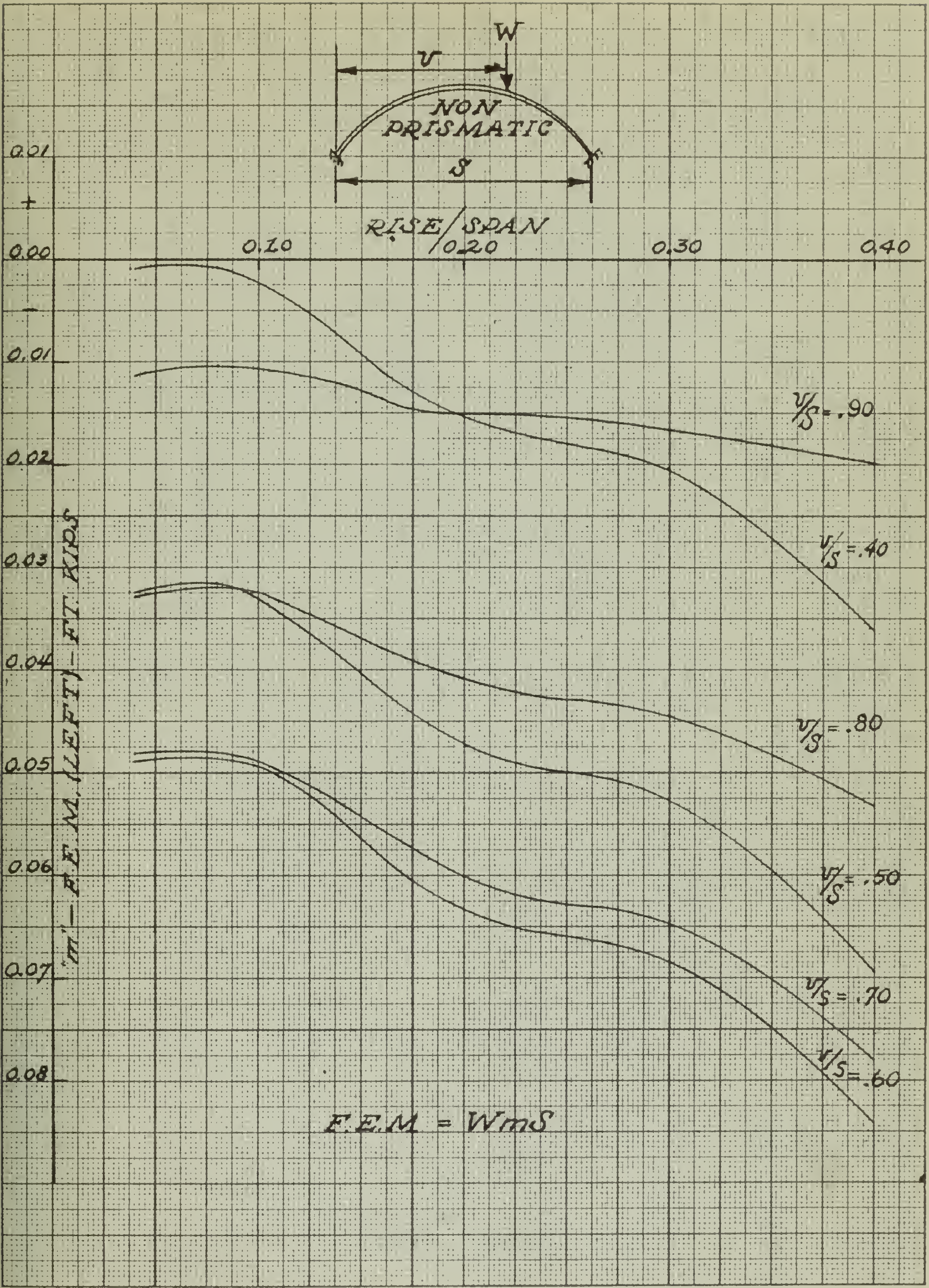
CURVES FOR NON-PRISMATIC ARCH

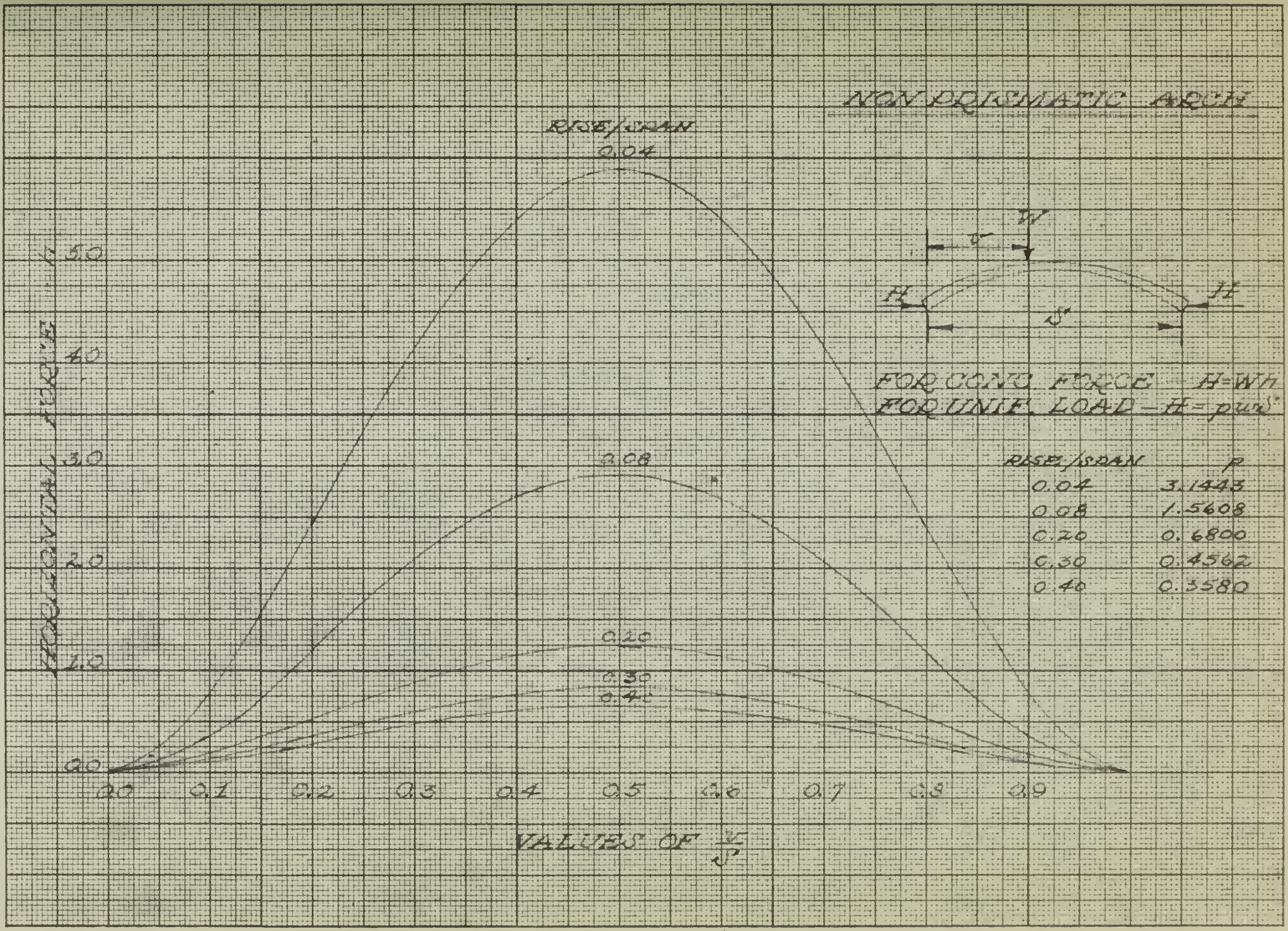
$I_x = I_c \sec \theta$

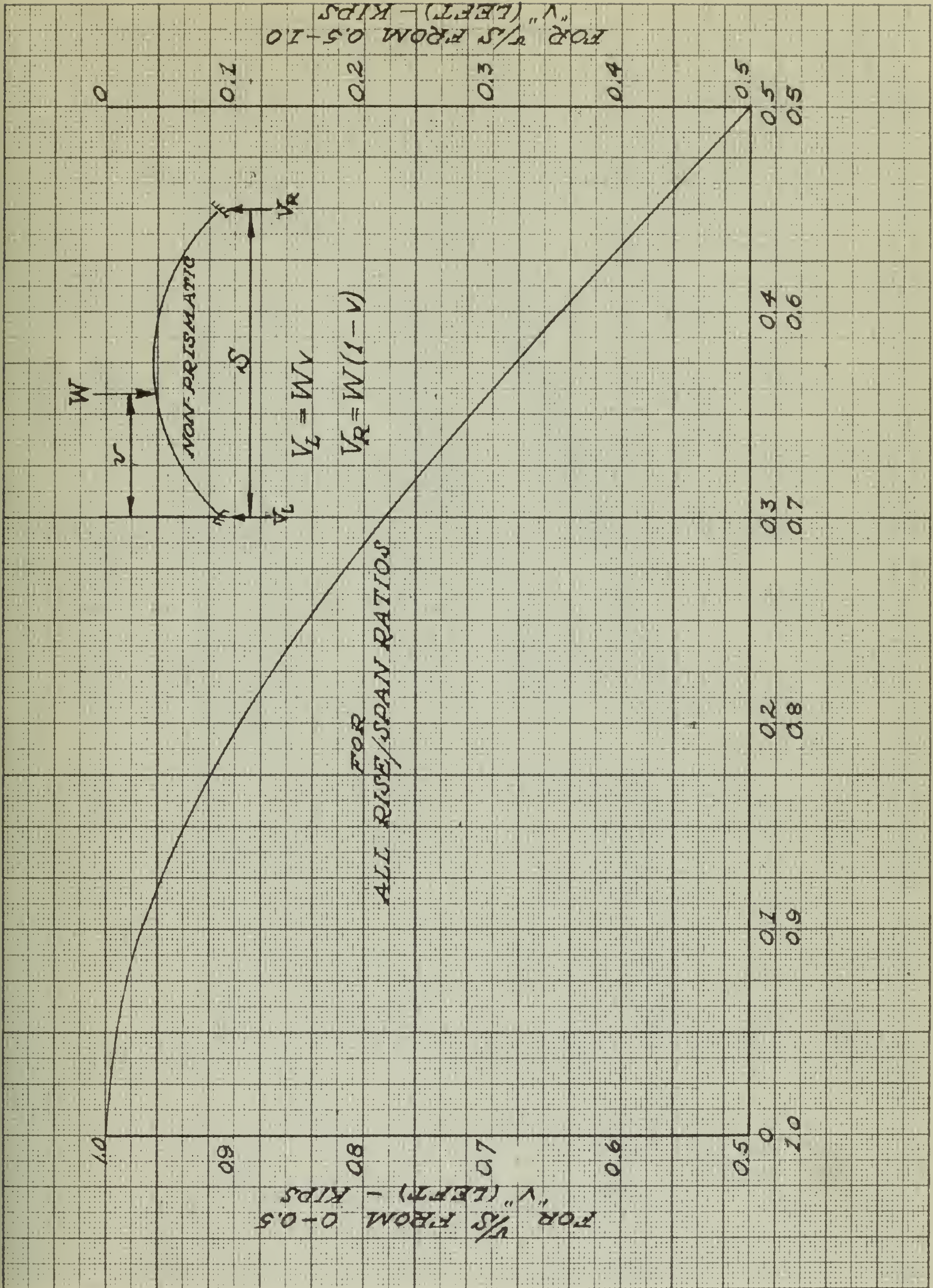

NON PRISMATIC ARCH
W
v
S
FOR CONC. LOAD M = mSW
FOR UNIF. LOAD M = mwS²
FIXED END MOMENT - m
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
(-) 0.01
(-) 0.02
(-) 0.03
(-) 0.04
(-) 0.05
(-) 0.06
(-) 0.07
(-) 0.08
0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
VALUES OF v/S
RISE/SPAN
0.04 - 0.08
0.20
0.30
0.40
RISE/SPAN u
0.04 (-)0.00272
0.08 (-)0.00072
0.20 (-)0.00994
0.30 (-)0.01340
0.40 (-)0.02247
FOR F.E.M. (LEFT) ENTER v/S
FOR F.E.M. (RIGHT) ENTER v/S

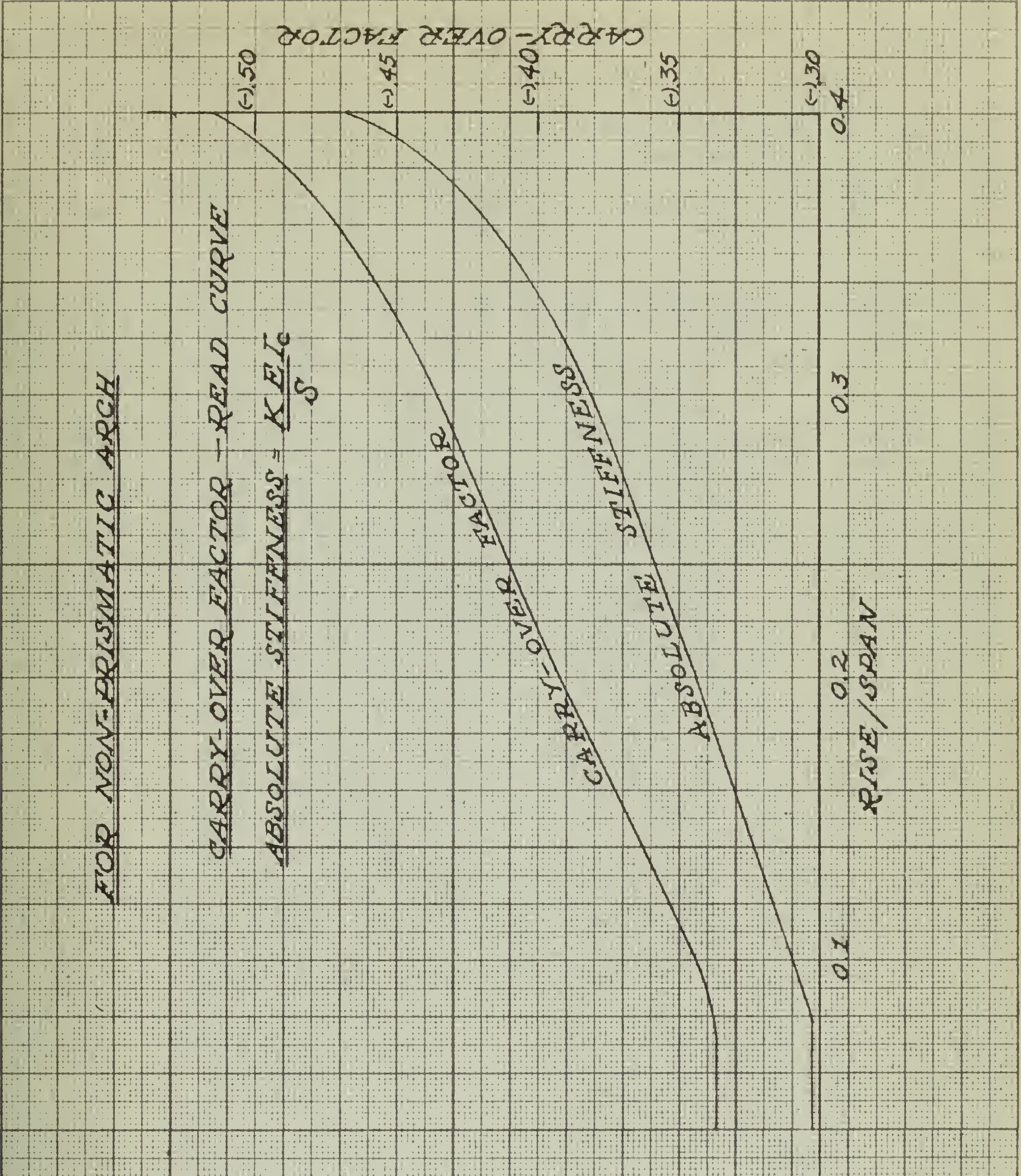

W
v
NON PRISMATIC
S
0.07
0.06
0.05
0.04
0.03
0.02
0.01
+
0.00
"m" - F.E.M. (LEFT) - FT. KIPS
v/S = .10
v/S = .20
v/S = .30
0.10
0.20
0.30
0.40
RISE / SPAN

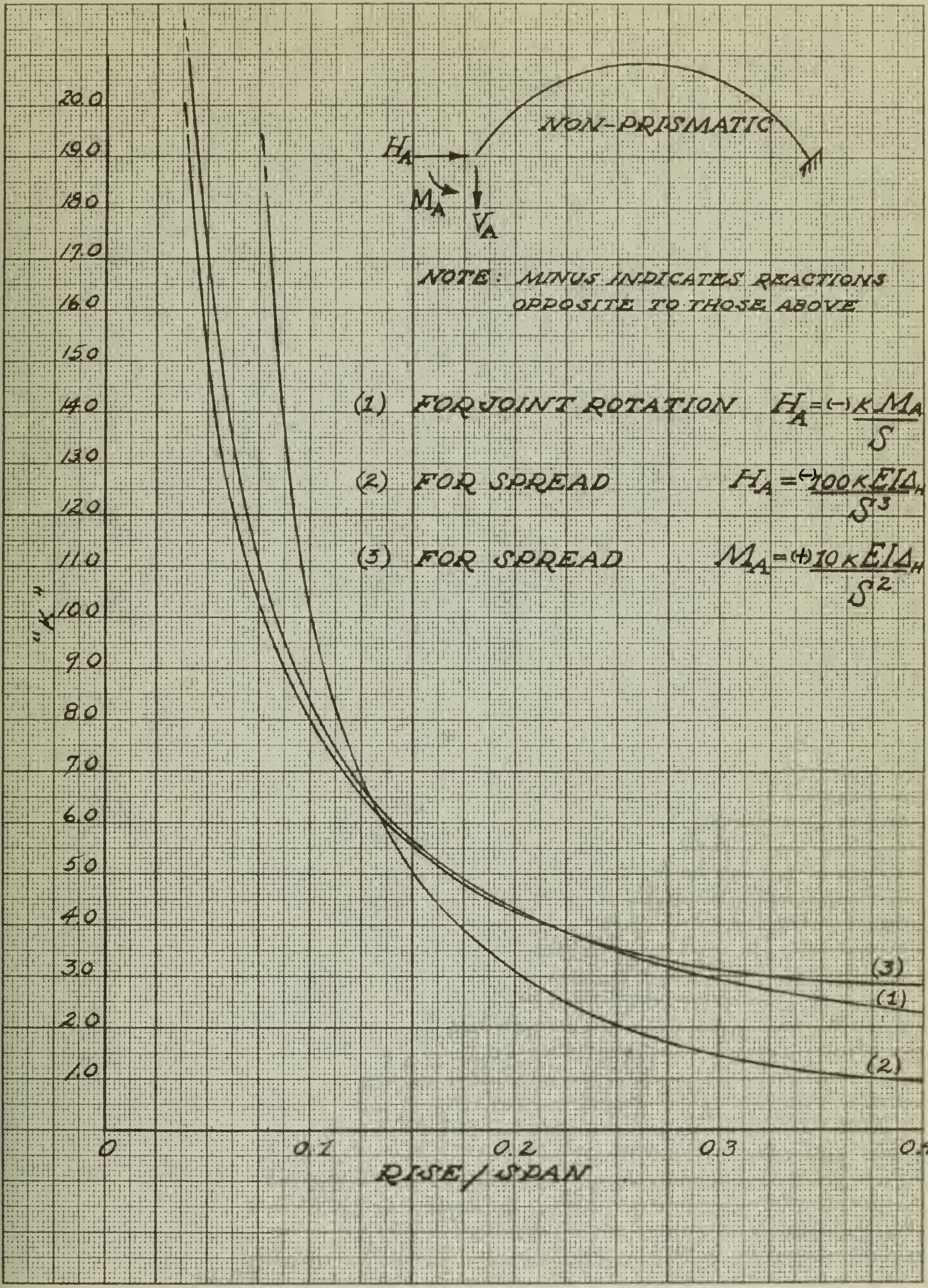
F.E.M. = WmS


W
v
NON PRISMATIC
S
RISE/SPAN
0.10
0.20
0.30
0.40
0.01
+
0.00
−
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
"m" − F.E.M. (LEFT) − FT. KIPS
v/S = .90
v/S = .40
v/S = .80
v/S = .50
v/S = .70
v/S = .60
F.E.M = WmS


NON PRISMATIC ARCH
RISE/SPAN
0.04
0.08
0.20
0.30
0.40
W
v
H
H
S
FOR CONC FORCE H=Wh
FOR UNIF. LOAD – H=pwS²
RISE/SPAN p
0.04 3.1443
0.08 1.5608
0.20 0.6800
0.30 0.4562
0.40 0.3580
HORIZONTAL FORCE h
0.0
1.0
2.0
3.0
4.0
5.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
VALUES OF $\frac{v}{S}$


FOR v/S FROM 0.5-1.0
"V" (LEFT) - KIPS
0
0.1
0.2
0.3
0.4
0.5
0.5
0.5
0.4
0.6
0.3
0.7
0.2
0.8
0.1
0.9
0
1.0
1.0
0.9
0.8
0.7
0.6
0.5
FOR v/S FROM 0-0.5
"V" (LEFT) - KIPS
W
v
NON-PRISMATIC
S
V_L
V_R
$V_L = Wv$
$V_R = W(1-v)$
FOR
ALL RISE/SPAN RATIOS


FOR NON-PRISMATIC ARCH
CARRY-OVER FACTOR - READ CURVE
ABSOLUTE STIFFNESS = $\frac{K E I_c}{S}$
CARRY-OVER FACTOR
ABSOLUTE STIFFNESS
RISE/SPAN
0
0.1
0.2
0.3
0.4
ABSOLUTE STIFFNESS "K"
9.0
10.0
11.0
12.0
13.0
CARRY-OVER FACTOR
(-).30
(-).35
(-).40
(-).45
(-).50


NON-PRISMATIC
H_A
M_A
V_A
NOTE: MINUS INDICATES REACTIONS OPPOSITE TO THOSE ABOVE
(1) FOR JOINT ROTATION $H_A = (-)\dfrac{K M_A}{S}$
(2) FOR SPREAD $H_A = (-)\dfrac{100 K E I \Delta_H}{S^3}$
(3) FOR SPREAD $M_A = (+)\dfrac{10 K E I \Delta_H}{S^2}$
"K"
20.0
19.0
18.0
17.0
16.0
15.0
14.0
13.0
12.0
11.0
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
0.1
0.2
0.3
0.4
RISE / SPAN
(3)
(1)
(2)

RESULTS FOR NON-PRISMATIC ARCH
BASED ON A UNIT SPAN

TABULATION OF CONSTANTS FOR A NON-PRISMATIC MEMBER BASED ON A UNIT SPAN

Rise/Span = 0.04

Load Point	Fem (Left) Ft.KIPS	Fem (Right) Ft.KIPS	V (Left)	H (Thrust)
1	+.060585 S	-.011415 S	.972000	.764589
2	+.063069 S	-.032931 S	.896000	2.431799
3	+.035853 S	-.048147 S	.784000	4.162677
4	-.000922 S	-.048922 S	.648000	5.427620
5	-.032446 S	-.032446 S	.500000	5.896671
Rise/Span = 0.08				
1	+.061564 S	-.010436 S	.972000	.382590
2	+.063975 S	-.032025 S	.896000	1.194396
3	+.035576 S	-.048425 S	.784000	2.078643
4	-.000846 S	-.048846 S	.648000	2.702132
5	-.031606 S	-.031606 S	.500000	2.921515
Rise/Span = 0.20				
1	+.058825 S	-.013175 S	.972000	.175661
2	+.055210 S	-.040790 S	.896000	.523908
3	+.023923 S	-.060077 S	.784000	.894297
4	-.015405 S	-.063405 S	.648000	1.153284
5	-.047449 S	-.047449 S	.500000	1.253049
Rise/Span = 0.30				
1	+.055303 S	-.016697 S	.972000	.120171
2	+.051587 S	-.044413 S	.896000	.357347
3	+.019117 S	-.064883 S	.784000	.598050
4	-.020492 S	-.068492 S	.648000	.759252
5	-.052644 S	-.052644 S	.500000	.830758
Rise/Span = 0.40				
1	+.052084 S	-.019916 S	.972000	.096968
2	+.042737 S	-.053263 S	.896000	.285925
3	+.005919 S	-.078081 S	.784000	.473109
4	-.036356 S	-.084356 S	.648000	.604816
5	-.069363 S	-.069363 S	.500000	.651056

Tabulation of Results for Non-Prismatic Arch

Rise/Span	Absolute Stiffness	Carry-over Factor
0.04	9.051219 EI_c/S	(-) .3371054
0.08	9.062642 EI_c/S	(-) .3379414
0.20	9.879728 EI_c/S	(-) .3926968
0.30	10.679830 EI_c/S	(-) .4555514
0.40	12.350376 EI_c/S	(-) .5141846

With Joint A free to rotate, and a moment of M_a ft.kips applied at A, the following thrusts and shears are induced at A.

Rise/Span	Thrust (KIPS)	Vertical Shear (KIPS)
0.04	20.900806 EI_cM_a/S	.6626946 EI_cM_a/S
0.08	10.421282 EI_cM_a/S	.6620586 EI_cM_a/S
0.20	4.324337 EI_cM_a/S	.6073012 EI_cM_a/S
0.30	2.916752 EI_cM_a/S	.5644486 EI_cM_a/S
0.40	2.267353 EI_cM_a/S	.4958152 EI_cM_a/S

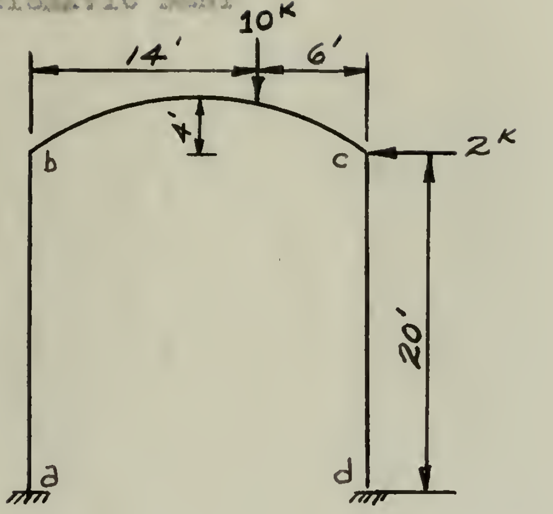
When a spread of ΔH feet occurs, the following moments, shears and thrusts occur.

Rise/Span	Thrust (KIPS)	Moment (Ft.KIPS)	Shear
0.04	7065.0562 $EI_c\Delta H/S^3$	(-)189.17764 $EI_c\Delta H/S^2$	0
0.08	1761.8736 $EI_c\Delta H/S^3$	(-) 94.44436 $EI_c\Delta H/S^2$	0
0.20	310.4368 $EI_c\Delta H/S^3$	(-) 42.72352 $EI_c\Delta H/S^2$	0
0.30	144.9938 $EI_c\Delta H/S^3$	(-) 31.00462 $EI_c\Delta H/S^2$	0
0.40	93.9058 $EI_c\Delta H/S^3$	(-) 28.00266 $EI_c\Delta H/S^2$	0

APPLICATION OF RESULTS

PRISMATIC ARCH BENT

FOR A PRISMATIC BENT



Rise = 4'

$\frac{\text{Rise}}{\text{Span}} = \frac{4}{20} = .2$

From Curves: M^f_{bc} = (-).0465 x 10^kx20 = (-) 9.30 ft.k.

M^f_{cb} = (+).0306 x 10 x 20 = (+) 6.12 ft.k.

H_b = .820 x 10 = 8.2^k ⟶

H_c = .820 x 10 = 8.2^k ⟵

with a positive moment indicating tension on the top of member. In moment distribution a moment that tends to rotate the joint clockwise is considered positive, therefore

M^f_{bc} = (-) 9.30 ft. k. M^f_{cb} = (-) 6.12 ft.k.

The absolute stiffness from page 57 is 7.6950 $\frac{EI}{L}$ = .38479 EI

Use .3848 EI

The carry over factor is (-).3489

The absolute stiffness of the prismatic column members is $\frac{4EI}{L}$ = .2EI and the carry over factor is (+).500.

FOR PRISMATIC BENT

Assuming rotation of the joints only, solve for the moments at all joints by moment distribution.

Joint	A	B				C		D
Member	AB	BA	BC			CB	CD	DC
K	.2	.2	.3848	BALANCING MOMENTS		.3848	.2	.2
$\frac{K}{\Sigma K}$	0	.342	.658			.658	.342	0
COF	0	.5	-.3489	-.3489		-.3489	+.5	0
			-9.30			-6.12		
	+1.59	+3.18	+6.12	+6.12		-2.14		
					+5.44	+5.44	+2.82	+1.41
			-1.90					
	+0.33	+0.65	+1.25	+1.25		-0.44		
			-0.10		+0.29	+0.29	+0.15	+ .08
	+0.02	+0.03	+0.07	+0.07		-0.02		
					+0.01	+0.01	+0.01	
M_F	+1.94	+3.86	-3.86			-2.98	+2.98	+1.49
Bal Mom				+7.44	+5.74			

A record of the balancing moments taken by the arch must be kept in order that the horizontal restraining force at the ends of the arched member may be determined. The magnitude of this restraining force depends on the properties of the arch under consideration.

The required horizontal restraining forces at the ends of the arch are made up of the following components:

(1) That force necessary to restrain the horizontal thrust at the ends of the arch caused by a load on the arch or caused by the tendency of the arch to spread.

(2) That force necessary to restrain the horizontal thrust at the ends of the arch as caused by the balancing moments.

(3) That force necessary to balance the shears in the column members.

(4) That force necessary to prevent sidesway when a horizontal load is applied to the bent.

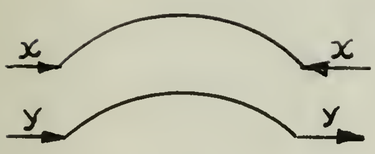
Using the sign convention of moment distribution, a positive balancing moment at the left springing causes horizontal restraining forces at the two ends of the arch acting in directions away from each other. A positive balancing moment at the right springing causes equal horizontal restraining forces acting in directions toward each other.

FOR PRISMATIC BENT

The artificial joint restraints (AJR) required to prevent the original bent from deflecting due to sidesway or spread are equal to the horizontal restraining forces at the ends of the arch. ($H = \frac{4.2168}{3} = .21084$ See Page 57)

HRF Caused by:	HRF_l $\xrightarrow{+}$	HRF_r
(a) Load on Arch	(+)8.20	(−)8.20
(b) Balancing Moment at B (−0.21084)(7.44)	(−)1.57	(+)1.57
(c) Balancing Moment at C (0.21084)(5.74)	(+)1.21	(−)1.21
(d) Shear in Member AB $\frac{(1.94 + 3.86)}{20}$	(+)0.29	
(e) Shear in Member CD $\frac{(1.49 + 2.98)}{20}$		(+)0.22
(f) 2 Kip Load at C		(+)2.00
	AJR_l (+)8.13	AJR_r (−)5.62

Let x = AJR resisting spread
Let y = AJR resisting sidesway
then

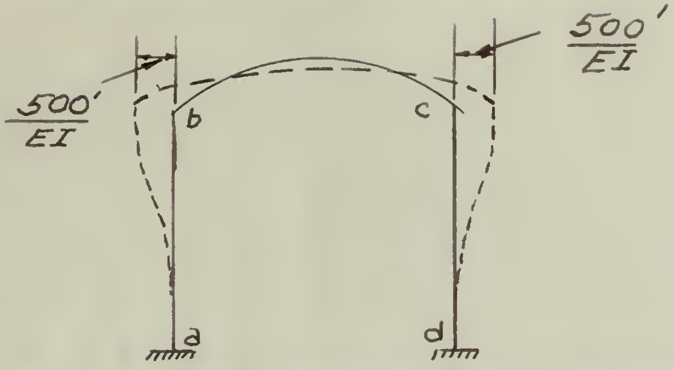


$$x + y = 8.13$$
$$x - y = 5.62$$
$$2x = 13.75$$
$$x = 6.875^k$$
$$y = 1.255^k$$

Thus the AJR_{sp} required to resist spread = 6.875^k → ←

and the AJR_{ss} required to resist sidesway = 1.255^k →

Assume a spread of 1000/EI feet, allowing no joint rotation.



The following reactions at the ends of the arch result from the above spread of $\frac{1000'}{EI}$

From Page 57

$$H_b = 245.8247 \frac{EI\Delta H}{S^3} = \frac{245.8247\ EI}{(20)^3} \times \frac{1000}{EI} = 30.7281 \longleftarrow$$

$$H_c = 30.7281 \longrightarrow$$

$$M^f_{bc} = (+)32.45204 \frac{EI\Delta H}{S^2} = \frac{32.45204\ EI}{(20)^2} \times \frac{1000}{EI} = (+)81.1301$$

$$M^f_{cb} = (+)81.130$$

The above moments indicate tension on the top side, hence for moment distribution sign convention:

$M^f_{bc} = (+)81.1301$ ft.k. $\quad M^f_{cb} = (-)81.1301$ ft.k.

The moments induced in the ends of the column members are equal to $\frac{6EI\Delta H}{L^2}$

$$= \frac{6EI}{(20)^2}\left(\frac{500}{EI}\right) = 7.5 \text{ ft.kips}$$

and $M^f_{ab} = (-)7.5 = M^f_{ba}$

$M^f_{cd} = (+)7.5 = M^f_{dc}$

Solving for moments at the joints caused by the assumed spread

Joint	A	B		BALANCING MOMENTS		C		D
Member	AB	BA	BC			CB	CD	DC
K	.2	0.2	0.3848			0.3848	0.2	.2
$\frac{K}{\Sigma K}$	0	0.342	0.658			0.658	0.342	0
COF	0	+ 0.5	-0.3469	-0.3480		-0.3488	+0.5	0
FEM	- 7.50	- 7.50	+81.13			-81.13	+ 7.50	+ 7.50
	-12.59	-25.18	-43.45	-43.45		+16.90		
			-13.02		+37.33	+37.33	+19.40	+ 9.70
	+ 2.23	+ 4.45	+ 8.57	+ 8.57		- 2.99		
			- 0.65		+ 1.97	+ 1.97	+ 1.02	+ 0.51
	+ 0.12	+ 0.24	+ 0.45	+ 0.45		- 0.16		
					+ 0.11	+ 0.11	+ 0.05	+ 0.03
M_{sp}	-17.74	-27.99	+27.99			-27.97	+27.97	+17.74
Bal. Moms.				-39.43	+39.41			

The consistent joint force (CJF) required at B and C to maintain the assumed spread must equal the horizontal restraining forces at B and C.

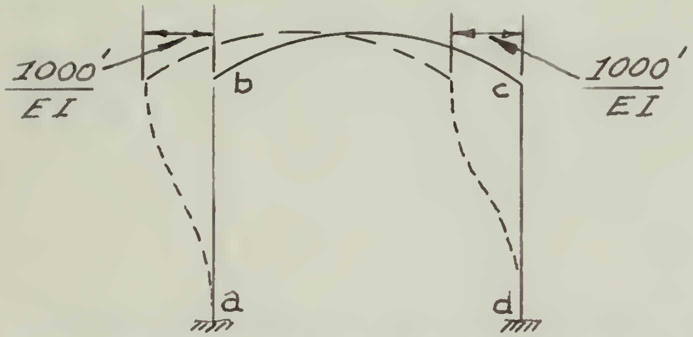
HRF Caused by:		HRF_L	$\xrightarrow{+}$	HRF_R
(1) Spread in the arch		-30.73		+30.73
(2) Balancing moment at joint B (0.21084) (39.43)		+ 8.31		- 8.31
(3) Balancing moment at joint C (0.21084) (39.41)		+ 8.31		- 8.31
(4) Shear in member AB	$\frac{17.74 + 27.99}{20} = \frac{45.73}{20}$	- 2.29		
(5) Shear in member BC	$\frac{17.74 + 27.99}{20} =$			+ 2.29
	CJF_{sp}	$\underleftarrow{-16.40^{k}}$		$\underrightarrow{16.40^{k}}$

Since a consistent joint force of 16.40^k at each end of the arch acting away from each other will produce the assumed spread of $\frac{1000}{EI}$ ft., and since an artificial joint restraint of 6.875 kips was required to restrain the original structure from spreading under the action of the 10 kip vertical load, the actual spread will cause moments at the various joints equal to

$$\frac{AJR_{sp}}{CJF_{sp}} \left(M_{sp} \text{ for } \frac{1000}{EI}\right) = \frac{6.875}{16.40} M_{sp} = .4192\, M_{sp}$$

SIDESWAY

Assume a sidesway of $\frac{1000}{EI}$ feet to the left.



The moments induced in the column members as a result of the assumed sidesway equals

$$\frac{6EI}{L^2} = \frac{6EI}{(20)^2}\,\frac{1000}{EI} = 15 \text{ fk.}$$

$M^f_{ab} = (-)15$ fk, $M^f_{ba} = -15$ fk,, $M^f_{cd} = -15$ fk., $M^f_{dc} = -15$ fk.

Solving for the moments at the joints caused by the assumed sidesway.

Joint	A	B		Balancing Moments		C		D
Member	AB	BA	BC			CB	CD	DC
K	0.2	0.2	0.3848			0.3848	0.2	0.2
$\frac{K}{\Sigma K}$	0	0.342	0.658			0.658	0.342	0
COF	0	+ 0.5	-0.3489	-0.3489		-0.3489	+ 0.5	0
FEM	-15.00	-15.00					-15.00	-15.00
	+ 2.57	+ 5.13	+ 9.87	+ 9.87		- 3.44		
			- 4.23		+12.13	+12.13	+ 6.31	+ 3.16
	+ 0.73	+ 1.45	+ 2.78	+ 2.78		- 0.97		
			- 0.22		+ 0.64	+ 0.64	+ 0.33	+ 0.17
	+ 0.04	+ 0.08	+ 0.14	+ 0.14				
M_{ss}	-11.66	- 8.34	+ 8.34			+ 8.36	- 8.36	-11.67
Bal. Moms.				+12.79	+12.77			

The consistent joint force (CJF) required at B and C to maintain the assumed sidesway must equal the horizontal restraining forces at B and C.

$\xrightarrow{+}$

HRF caused by:	HRF_L	HRF_R
(1) Balancing Moment at Joint B (0.21084)(12.79)	(-)2.69	(+)2.69
(2) Balancing Moment at Joint C (0.21084)(12.77)	(+)2.69	(-)2.69
(3) Shear in Member AB $\frac{11.66 + 8.34}{20}$	(-)1.00	
(4) Shear in Member CD $\frac{11.67 + 8.36}{20}$		(-)1.00
CJF_{ss}	(-)1.00^k ←	(-)1.00^k ←

Since a consistent joint force of 1.00 kips at each end of the arch acting toward the left will produce the assumed sidesway of $\frac{1000}{EI}$ feet, and since an artificial joint restraint of 1.255 kips was required to prevent sidesway in the original structure under the action of the 10 k vertical load, the actual sidesway will cause movements at the various joints equal to

$$\frac{AJR_{ss}}{CJF_{ss}} \left(M_{ss} \text{ for } \frac{1000}{EI}\right) = \frac{1.255}{1.00} M_{ss} = 1.255\, M_{ss}$$

TABULATION OF RESULTS

Factor	M_{ab}	M_{ba}	M_{bc}	M_{cb}	M_{cd}	M_{dc}
M_r	(+) 1.94	(+) 3.86	(-) 3.86	(-) 2.98	(+) 2.98	(+) 1.49
.4192 M_{sp}	(-)7.44	(-)11.73	(+)11.73	(-)11.73	(+)11.73	(+) 7.44
1.255 M_{ss}	(-)14.63	(-)10.47	(+)10.47	(+)10.49	(-)10.49	(-)14.65
Final Internal Moments	(-)20.13	(-)18.34	(+)18.34	(-) 4.22	(+) 4.22	(-) 5.72

$$H_a = \frac{20.13 + 18.34}{20} = 1.925^k \longrightarrow$$

$$H_d = \frac{(+)4.22\ (-)5.72}{20} = \frac{(-)1.50}{20} = .075 \longrightarrow$$

Cutting the Bent at A

$$V_a(20) + 20.13 - 60 - 40 = (-)5.72$$

$$V_a = \frac{(-)20.13 + 60 + 40\ (-)5.72}{20} = \frac{74.15}{20} = 3.7075 \uparrow$$

Cutting the Bent at B

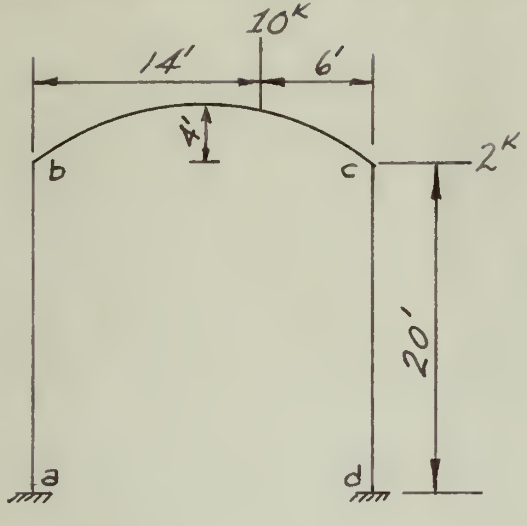
$$(-)V_d(20) + 5.72 - 40 + 140 = (-)\ 20.13$$

$$V_d = \frac{(-)20.13 - 140 + 40 - 5.72}{(-)20} = \frac{(-)125.85}{(-)20} = 6.2925 \uparrow$$

APPLICATION OF RESULTS

NON-PRISMATIC ARCH BENT

FOR A NON-PRISMATIC BENT ($I_x = I_c \sec\theta$)



The application of the method of moment distribution to the non-prismatic is the same as for the prismatic. This problem is worked primarily to show the relative values of moments and reactions developed under the same loading on a bent whose rise to span ratio and overall dimensions are the same as for the prismatic arch bent but whose I_{crown} is equal to the $I_{prismatic}$ multiplied by $\cos\theta_a = I \times .72415$

From Curves:

$M^f_{bc} = (-).0601 \times 10^k \times 20' = (-)12.02$ ft.k.

$M^f_{cb} = (+).0239 \times 10 \times 20 = (+)4.78$ ft.k.

$H_b = .8943 \times 10 = 8.943$ k →

$H_c = .8943 \times 10 = 8.943$ k ←

For Moment Distribution Sign Convention

$M^f_{bc} = (-)12.02$ ft.k. $M^f_{cb} = (-)4.78$ ft.k.

Absolute Stiffness of:

Arch (Page 66) $9.87973\frac{EI_c}{S} = \frac{9.87973(.72415)EI}{20} = .35772\ EI$

Column Members $\frac{4EI}{L} = 0.2\ EI$

Carry-over Factor of:

Arch (Page 66) = (-).3927 Column Members = (+)0.50

Assuming rotation of joints only,

Joint	A	B		Balancing Moments		C		D
Member	AB	BA	BC			CB	CD	DC
K	0.2	0.2	.358			0.358	0.2	0.2
$\frac{K}{\Sigma K}$	0	0.358	0.642			0.642	0.358	0
COF	0	+ 0.5	-0.393	-0.393		-0.393	+ 0.5	0
FEM			-12.02			- 4.78		
	+ 2.15	+ 4.30	+ 7.72	+ 7.72		- 3.03		
			- 1.97		+ 5.01	+ 5.01	+ 2.80	+ 1.40
	+ 0.36	+ 0.71	+ 1.26	+ 1.26		- 0.50		
			- 0.13		+ 0.32	+ 0.32	+ 0.18	+ 0.09
	+ 0.03	+ 0.05	+ 0.08	+ 0.08		- 0.03		
					+ 0.02	+ 0.02	+ 0.01	
M_r	+ 2.54	+ 5.06	- 5.06			- 2.99	+ 2.99	+ 1.49
Bal. Mom.				+ 9.06	+ 5.35			

The artificial joint restraints = ARF's:

ARF Caused by:	$ARF_L \xrightarrow{+}$	ARF_R
(1) Load on Arch	(+)8.94	(-)8.94
(2) Balancing Moment at B (.21622)(9.06)	(-)1.96	(+)1.96
(3) Balancing Moment at C (.21622)(5.35)	(+)1.16	(-)1.16
(4) Shear in AB $\frac{2.54 + 5.06}{20}$	(+)0.38	
(5) Shear in CD $\frac{2.99 + 1.49}{20}$		(+)0.22
(6) 2 kip load at C		(+)2.00
	(+)8.52	(-)5.92

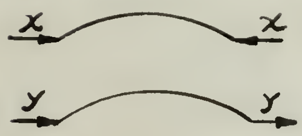
Let x = AJR resisting spread $x + y = 8.52$

y = AJR resisting sidesway $x - y = 5.92$

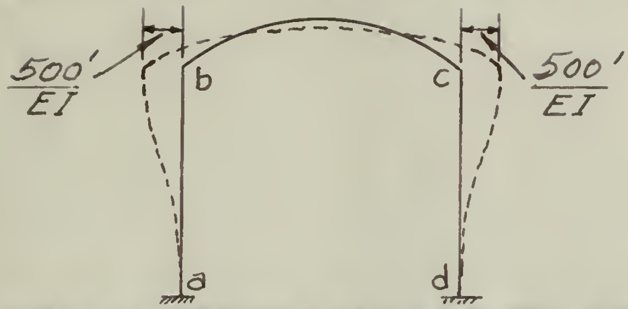
$2x = 14.44$

$x = 7.22^k$ = AJR resisting spread (sp)

$y = 1.30k$ = AJR resisting sidesway (ss)



Assuming a spread of 1000/EI feet, allowing no joint rotation.



The following reactions result at the arch ends.

From Page 66

$$H_b = 310.4368 \frac{EI_c \Delta H}{S^3} = \frac{310.4368}{(20)^3} EI \times (.72415) \times \frac{1000}{EI} = 28.1006 \longleftarrow$$

$$H_c = 28.1006 \longrightarrow$$

$$M^f_{bc} = 42.7235 \frac{EI_c \Delta H}{S^2} = \frac{42.7235}{(20)^2} EI(.72415) \times \frac{1000}{EI} = (+)77.345$$

$$M^f_{cb} = (+)77.345$$

For Moment Distribution Sign Convention

$M^f_{bc} = (+)77.345$ $\qquad$ $M^f_{cb} = (-)77.345$

The moments induced at the ends of the column members are equal to $\frac{6EI\Delta H}{L^2}$

$$= \frac{6EI\ (500)}{(20)^2\ EI} = 7.5 \text{ ft.kips}$$

and $M^f_{ab} = (-)7.5$ ft.kips $= M^f_{ba}$

$M^f_{cd} = (+)7.5$ ft. kips $= M^f_{dc}$

Solving for moments at the joints caused by the assumed spread

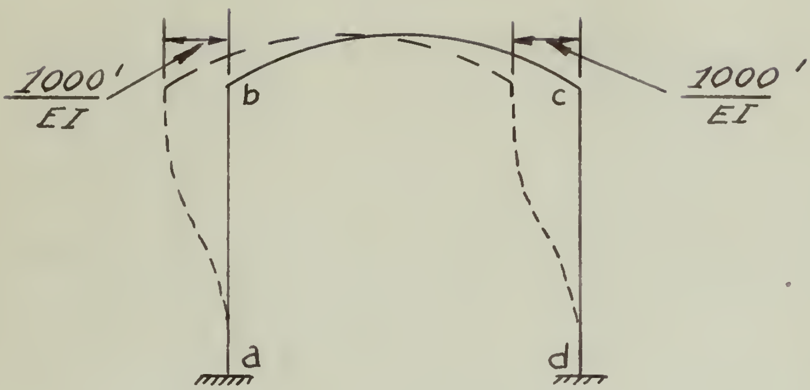
Joint	A	B		Balancing Moments		C		D
Member	AB	BA	BC			CB	CD	DC
K	0.2	0.2	0.358			0.358	0.2	0.2
$\frac{K}{\Sigma K}$	0	0.358	0.642			0.642	0.358	0
COF	0	+ 0.5	-0.393	-0.393		-0.393	+ 0.5	0
FEM	- 7.50	- 7.50	+77.35			-77.35	+ 7.50	+ 7.50
	-12.51	-25.01	-44.84	-44.84		+17.62		
			-13.18		+33.53	+33.53	+18.70	+ 9.35
	+ 2.36	+ 4.72	+ 8.46	+ 8.46		- 3.32		
			- 0.84		+ 2.13	+ 2.13	+ 1.19	+ 0.60
	+ 0.15	+ 0.30	+ 0.54	+ 0.54		- 0.21		
			- 0.05		+ 0.13	+ 0.13	+ 0.08	+ 0.04
	+ 0.01	+ 0.02	+ 0.03	+ 0.03				
Σ_T	-17.49	-27.47	+27.47			-27.47	+27.47	+17.49
Bal. Mom.				-35.81	+35.79			

The CJF_{sp} required at B and C = Σ HRF's

HRF Caused by:	$HRF_L \xrightarrow{+}$	HRF_R
(1) Spread in the arch	(-)28.10	(+)28.10
(2) Balancing Moment at B (.21622)(35.81)	(+) 7.74	(-) 7.74
(3) Balancing Moment at C (.21622)(35.79)	(+) 7.74	(-) 7.74
(4) Shear in Member AB $\frac{(17.49+27.47)}{20} = \frac{44.96}{20}$	(-) 2.25	
(5) Shear in Member BC		(+) 2.25
CJF_{sp} =	(-)14.87	(+)14.87
	←	→

$$\frac{AJM_{sp}}{CJF_{sp}} \left(M_{sp} \text{ for } \frac{1000}{EI}\right) = \frac{7.22}{14.87} = .4855\ M_{sp}$$

SIDESWAY



The moments induced in the column members as a result of the assumed sidesway equals

$$\frac{6EI}{L^2} = \frac{6EI}{(20)^2} \times \frac{1000}{EI} = 15 \text{ f.k.}$$

$M^f_{ab} = (-)15$ fk, $M^f_{ba} = (-)\ 15$ fk, $M^f_{cd} = (-)15$ fk, $M^f_{dc} = (-)15$ fk

Solving for the moments at the joints caused by the assumed sidesway.

Joint	A	B				C		D
Member	AB	BA	BC	BALANCING MOMENTS		CB	CD	DC
K	0.2	0.2	0.358			0.358	0.2	0.2
$\frac{K}{\sum K}$	0	0.358	0.642			0.642	0.358	0
COF	0	+ 0.5	-0.393	-0.393		-0.393	+ 0.5	0
FEM	-15.00	-15.00					-15.00	-15.00
	+ 2.69	+ 5.37	+ 9.63	+ 9.63		- 3.78		
			- 4.74		+12.06	+12.06	+ 6.72	+ 3.36
	+ 0.85	+ 1.70	+ 3.04	+ 3.04		- 1.19		
			- 0.30		+ 0.76	+ 0.76	+ 0.43	+ 0.22
	+ 0.06	+ 0.11	+ 0.19	+ 0.19		- 0.07		
					+ 0.04	+ 0.04	+ 0.05	+ 0.02
M_r	-11.40	- 7.82	+ 7.82			+ 7.82	- 7.82	-11.40
Bal. Mom.				+12.86	+12.86			

The CJF at B and C = $\sum$ HRF's

HRF Caused by:	HRF_L	$\xrightarrow{+}$ HRF_R
(1) Balancing Moment at B (.21622)(12.86)	(-) 2.78	(+) 2.78
(2) Balancing Moment at C (.21622)(12.86)	(+) 2.78	(-) 2.78
(3) Shear in AB $\frac{11.40 + 7.82}{20}$	(-) 0.96	
(4) Shear in BC		(-) 0.96
CJF_{ss}	$(-)\ \underset{\longleftarrow}{0.96^{k}}$	$(-)\ \underset{\longleftarrow}{0.96^{k}}$

$$\frac{AJR_{ss}}{CJF_{ss}}\ (M_{ss}\ \text{for}\ \frac{1000}{EI}) = \frac{1.30}{.96} M_{ss} = 1.354\ M_{ss}$$

TABULATION OF RESULTS:

Factor	M_{ab}	M_{ba}	M_{bc}	M_{cb}	M_{cd}	M_{oc}
M_r	(+) 2.54	(+) 5.06	(-) 5.06	(-) 2.99	(+) 2.99	(+) 1.49
.4855 M_{sp}	(-) 8.48	(-)13.34	(+)13.34	(-)13.34	(+)13.34	(+) 8.48
1.354 M_{ss}	(-)15.44	(-)10.59	(+)10.59	(+)10.59	(-)10.59	(-)15.44
Final Internal Moments	(-)21.38	(-)18.87	(+)18.87	(-) 5.74	(+) 5.74	(-) 5.47

$$H_a = \frac{21.38 + 18.87}{20} = \frac{40.25}{20} = 2.013^{\,k} \longrightarrow$$

$$H_d = \frac{(+)5.74\ (-)5.47}{20} = \frac{(+).27}{20} = .013^{\,k} \longleftarrow$$

Cutting the Bent at "A"

$$V_a(20) + 21.38 - 60 - 40 = (-)5.47$$

$$V_a = \frac{(-)21.38 + 60 + 40 - 5.47}{20} = \frac{73.15}{20} = 3.6575^{\,k} \uparrow$$

Cutting the Bent at "B"

$$-V_d(20) + 5.47 - 40 + 140 = (-)21.38$$

$$V_d = \frac{(-)21.38 - 140 + 40 - 5.47}{(-)20} = 6.3425^{\,k} \uparrow$$

APPENDIX A

Prismatic Arch

Sample calculations for determining:

(1) Reactions at Neutral Point.

(2) Fixed End Moment.

(3) Carry-Over Factor.

(4) Horizontal Thrust Due to Rotation of the Joint.

(5) Absolute Stiffness.

(6) Effect of Spread.

DETERMINATION OF δ_{vv}, δ_{hh}, and $\alpha_{\theta\theta}$ AT NEUTRAL POINT FOR PRISMATIC ARCH

From Formulas Previously Derived

$\delta_{vv} = \frac{R}{2}\left[\widehat{L}R - S(R-r)\right]$ $\delta_{hh} = \frac{R^2S^2}{\widehat{L}} - \frac{R}{2}\left[\widehat{L}R + S(R-r)\right]$ $\alpha_{\theta\theta} = \widehat{L}$

Rise/Span = .20 R = .725' R/2 = .3625' $\widehat{L}$ = 1.103479'

$\widehat{L}R-S(R-r)$	$\widehat{L}R+S(R-r)$	$\frac{R^2S^2}{\widehat{L}}$	δ_{vv}	δ_{hh}	$\alpha_{\theta\theta}$
.27524775	1.32524775	.47655439	(+).09977731	(-).00410584	(+)1.103479

DETERMINATION OF V_o, H_o and θ at NEUTRAL POINT FOR PRISMATIC ARCH

Rise/Span = .20

A = 43°36'11.5" = .76102 Radians CosA = .72415 SinA = .68965

Load Point	$\sin\theta_{1p}$	θ_{1p}	$\cos\theta_{1p}$	θ_{1p} Radians
1	0	0	1.00000	0
2	.13793	7°55'41.4"	.99044	.13837
3	.27586	16°00'47.2"	.96120	.27948
4	.41379	24°26'35.6"	.91037	.42662
5	.55172	33°29' 7.2"	.83403	.58443

From Formula Previously Derived

$$V = \frac{R}{4}\left[\widehat{L}R - S(R-r)\right] - \frac{R^3}{2}\left[\theta_{1p} - \sin\theta_{1p}\cos\theta_{1p}\right] + R(X_{1p})(R-r) - R^2(X_{1p})\cos\theta_{1p}$$

Load Point	X_{1p}	$R/4\left[\widehat{L}R-S(R-r)\right]$	$\frac{R^3}{2}(\theta_{1p}-\sin\theta_{1p}\cos\theta_{1p})$	$RX_{1p}(R-r)$	$R^2X_{1p}(\cos\theta_{1p})$	ΔV
1	0	.04988865	0	0	0	.04988865
2	.1	.04988865	.00033535	.03806250	.05206000	.03555530
3	.2	.04988865	.00272052	.07612500	.10104615	.02223893
4	.3	.04988865	.00951171	.11418750	.14355397	.01101047
5	.4	.04988865	.02368013	.15225000	.17535431	.00310365

DETERMINATION OF V_o, H_o and ϕ AT NEUTRAL POINT FOR PRISMATIC ARCH (Cont.)

$V_o = -\frac{\Delta V}{\delta_{vv}}$ $\delta_{vv} = .09977731$

Load Point	ΔV	V_o
1	.04988865	(-).5000000
2	.03555580	(-).3563515
3	.02223898	(-).2228860
4	.01101047	(-).1103504
5	.00310365	(-).0311058

From Formula Previously Derived

$$\Delta H = \frac{RS^2}{8} + \frac{R(X_{1p})^2}{2} + R(R-r)\left(\frac{RS}{\widehat{L}}\right) - R^2 Cos\theta_{1p}\left(\frac{RS}{\widehat{L}}\right) - (R)(X_{1p})\theta_{1p}\left(\frac{RS}{\widehat{L}}\right)$$

Load Point	X_{1p}	$\frac{RS}{\widehat{L}}$	$\frac{RS^2}{8}$	$\frac{R(X_{1p})^2}{2}$	$R(R-r)\frac{RS}{\widehat{L}}$	$R^2Cos\theta_{1p}\left(\frac{RS}{\widehat{L}}\right)$
1	0	.65701295	.090625	0	.25007555	.34534243
2	.1	.65701295	.090625	.003625	.25007555	.34204094
3	.2	.65701295	.090625	.014500	.25007555	.33194311
4	.3	.65701295	.090625	.032625	.25007555	.31438937
5	.4	.65701295	.090625	.058000	.25007555	.28802594

X_{1p}	$R(X_{1p})(\theta_{1p})\frac{RS}{\widehat{L}}$	ΔH	δ_{hh}	H_o
0	0	(-).00464186	(-).00410584	(-)1.1305555
.1	.00659104	(-).00430643	(-).00410584	(-)1.0488539
.2	.02662510	(-).00336775	(-).00410584	(-) .8202331
.3	.06096413	(-).00202795	(-).00410584	(-) .4939184
.4	.11135364	(-).00067903	(-).00410584	(-) .1653815

From Formula Previously Derived

$$\phi = R(R-r) + R^2(Cos\theta_{1p}) - \frac{(X_{1p})\widehat{L}}{2} + R(X_{1p})\theta_{1p}$$

$$M_o = -\frac{\phi}{\alpha_{mm}}$$

DETERMINATION OF V_o, H_o and θ AT NEUTRAL POINT FOR PRISMATIC ARCH (Cont.)

Load Point	λ_{1p}	$R(R-r)$	$R^2 \mathrm{Cos}\,\theta_{1p}$	$\frac{\lambda_{1p}\widehat{L}}{2}$	$R(\lambda_{1p})(\theta_{1p})$
1	0	.3806250	.5256250	0	0
2	.1	.3806250	.5206000	.05517395	.01003183
3	.2	.3806250	.5052307	.11034790	.04052460
4	.3	.3806250	.4785132	.16552185	.09278985
5	.4	.3806250	.4383870	.22069580	.16948470

Load Point	θ	M_o
1	.1450000	.1314026
2	.0948328	.0859398
3	.0547324	.0496452
4	.0251562	.0227972
5	.0065509	.0059356

DETERMINATION OF FIXED END MOMENT FOR PRISMATIC ARCH

$$M^f_l = V_o\ {}^S\!/_2 - H_o\left[r-(R-\bar{y})\right] + M_o$$

$$M^f_r = V_o\ {}^S\!/_2 - H_o\left[r-(R-\bar{y})\right] + M_o + 1\left({}^S\!/_2 - X_{1p}\right)$$

X_{1p}	$V_o\ {}^S\!/_2$	$H_o\left[r-(R-\bar{y})\right]$	M_o
0	(-.5000000)(-.5)	(-1.1305555)(-.1320130)	(-.1514026)
.1	(-.3563515)(-.5)	(-1.0495539)(-.1320130)	(-.0559393)
.2	(-.2225860)(-.5)	(-0.8202331)(-.1320130)	(-.0496452)
.3	(-.1103504)(-.5)	(-0.4939184)(-.1320130)	(-.0287972)
.4	(-.0311058)(-.5)	(-0.1653815)(-.1320130)	(-.0059366)

M^f_l	M^f_r
(-).030651	(-).030651
(-).046224	(-).002575
(-).046454	(+).030630
(-).032826	(+).056624
(-).012216	(+).056673

The Fixed End Moments shown above are caused by 1 kip loads placed at distances X_{1p} to the right of the crown. F.E.M. (left) as caused by a 1^k load to the right of the crown equals F.E.M. (right) as caused by a 1^k load placed at the corresponding point to the left of the crown. All F.E.M. values were tabulated on this basis.

DETERMINATION OF CARRY OVER FACTOR, THRUST, AND ABSOLUTE STIFFNESS (PRISMATIC)

Span Length = 1.0 Rise/Span = .20 R = .725' $\widehat{L}$ = 1.103479

Carry Over

(1) $EI\,\delta'_{vm} = EI\,\alpha'_{ev} = \widehat{L}\,S/_2 = .55173950$

(2) $EI\,\delta'_{hm} = EI\,\alpha'_{eh} = \widehat{L}\,[\text{Rise} - (R-\bar{y})]$

$= 1.103479\,[.2 - (.725 - .65701295)] = .14567352$

(3) $EI\,\alpha_{em} = \widehat{L} = 1.1034790$

(4) $EI\,\delta_{vv} = \frac{S^2\widehat{L}}{4} + R/_2[\widehat{L}R - S(R-r)]$

$= \frac{(1.103479)}{4} + \frac{.725}{2}\,[.725(1.103479) - 1(.725 - .2)] = .37564705$

(5) $EI\,\delta_{hh} = R/_2[\widehat{L}R + S(R-r)] + (R-r)^2\,\widehat{L} - 2R(R-r)\,S$

$= .43040233 + .30414640 - .761250 = +\,.02329873$

(6) $EI\,\delta_{vh} = EI\,\delta_{hv} = \widehat{L}\,S/_2\,[\text{Rise} - (R-\bar{y})] = S/_2(2) = .07283676$

$$D_2 = \frac{(EI)^2[(\delta'_{vm})(\delta_{hh}) - (\delta'_{hm})(\delta_{vh})]}{(EI)^2[(\delta_{hv})(\delta_{vh}) - (\delta_{vv})(\delta_{hh})]} = \frac{(1)(5) - (2)(6)}{(6)(6) - (4)(5)}$$

$$D_2 = \frac{.01285463 - .01061039}{.00530519 - .00875210} = -.6511470$$

$V_a = D_2 M_a = -.6511470 M_a$

$-M_b = -(1+D_2)(1)M_a = -(1-.651147)M_a = -.348853 M_a$

Carry Over Factor = -.348853

DETERMINATION OF CARRY OVER FACTOR, THRUST AND ABSOLUTE STIFFNESS (PRISMATIC)

Thrust

$$D_1 = \frac{(EI)^2[(\delta'_{vm})(\delta_{hv}) - (\delta'_{hm})(\delta_{vv})}{(EI)^2[(\delta_{hh})(\delta_{vv}) - (\delta_{hv})(\delta_{vh})} = \frac{(1)(6) - (2)(4)}{(5)(4) - (6)(6)}$$

$$= \frac{.04018692 - .05472183}{.00875210 - .00530519} = (-)4.216799$$

$$H_a = D_1 M_a = -4.216799\ M_a$$

Absolute Stiffness

From Formulas Previously Derived

When $\theta_a = 1$ $M_a = \frac{EI}{D_3}$ where $D_3 = (\alpha_{\theta\theta} + D_2\alpha'_{\theta v} + D_1\alpha'_{\theta h})EI$

$(EI)(\alpha_{\theta\theta}) = \hat{L} = 1.103479$

$(EI)(D_2)(\alpha'_{\theta v}) = (D_2)(\frac{\hat{L}S}{2}) = (-.651147)(.5517395) = -.35926352$

$(EI)(D_1)(\alpha'_{\theta h}) = (D_1)\left[\hat{L}(Rise-[R-\bar{y}])\right] = (-4.216799)(.14567352) =$

$= -.61427595$

$D_3 = 1.103479 - .35926352 - .61427595 = +.1299395$

Absolute Stiffness $= M_a = \frac{EI}{D_3} = 7.695889\ EI$

DETERMINATION OF MOMENT AND REACTIONS DUE TO THE EFFECT OF SPREAD (PRISMATIC)

Moment

From Formulas Previously Derived

$$M_a = \frac{EI\Delta H}{D_4} \quad \text{where } D_4 = EI\delta'_{hm} - \frac{(\delta'_{vm})(\delta_{hh})(EI)^2}{(\delta_{vh})(EI)}$$

(1) $EI\delta'_{hm} = .14567352$

(2) $(EI\delta'_{vm})(EI\delta_{hh}) = (.55173950)(.02329873) = .01285483$

(3) $EI\delta_{vh} = .07283676$

(4) $D_4 = (1) - \frac{(2)}{(3)} = .14567352 - \frac{.01285483}{.07283676} = -.0308147$

(5) $M_a = \frac{EI\Delta H}{(-).0308147} = -32.45204EI\Delta H$

Vertical Reaction

From Formulas Previously Derived

$V_a = 0$

Horizontal Reaction

From Formulas Previously Derived

$$H_a = \frac{EI\Delta H}{D_5} \quad \text{where } D_5 = EI\delta_{hh} - \frac{(\delta_{vh})(\delta'_{hm})(EI)^2}{EI\delta'_{vm}}$$

(1) $EI\delta_{hh} = .02329873$

(2) $(EI\delta_{vh})(EI\delta'_{hm}) = (.07283676)(.14567352) = .01061059$

(3) $EI\delta'_{vm} = .55173950$

(4) $D_5 = (1) - \frac{(2)}{(3)} = +.00406794$

(5) $H_a = \frac{EI\Delta H}{.00406794} = (+)245.8247EI\Delta H$

APPENDIX B

NON-PRISMATIC ARCH

Sample Calculations for Determining:

(1) Reactions at Neutral Point.

(2) Fixed End Moment.

(3) Carry-Over Factor.

(4) Horizontal Thrust Due to Rotation of
the Joint.

(5) Absolute Stiffness.

(6) Effect of Spread.

DETERMINATION OF δ_{vv}, δ_{hh} and α_{mm} AT NEUTRAL POINT FOR NON-PRISMATIC ARCH

From Formulas Previously Derived

$$\delta_{vv} = \frac{S^3}{12} \quad \delta_{vv} = \left[(\bar{y})^2 - R^2\right]S + \frac{S^3}{12} \qquad \alpha_{mm} = S$$

Rise/Span = .20 R = .725' R/2 = .3625' $\widehat{L}$ = 1.103479'

$$\delta_{vv} = .083333 \quad \delta_{hh} = .0032812 \quad \alpha_{mm} = 1.00000$$

DETERMINATION OF V_o, H_o and θ AT NEUTRAL POINT FOR NON-PRISMATIC ARCH

Rise/Span = .20

From Formula Previously Derived

$$V_o = -\frac{\Delta V}{\delta_{vv}} = -\left[.5 + 2\left(\frac{X_{1p}}{S}\right)^3 - 3/2\left(\frac{X_{1p}}{S}\right)\right]$$

Load Point	X_{1p}	$2\left(\frac{X_{1p}}{S}\right)^3$	$3/2\left(\frac{X_{1p}}{S}\right)$	V_o
1	0	-	-	.500
2	.1	.002	.15	.352
3	.2	.016	.30	.216
4	.3	.054	.45	.104
5	.4	.128	.60	.028

From Formula Previously Derived

$$\Delta H = (-)\left(\frac{R-r}{3}\right)^3 - \bar{y}\left(\frac{S^2}{6} + \frac{X_{1p}^2}{2}\right) + \frac{R^3}{3}\cos^3\theta_{1p} + \frac{R^2X_{1p}}{2}(\theta_{1p} + \sin\theta_{1p}\cos\theta_{1p})$$

For values $\sin\theta_{1p}$, θ_{1p}, $\cos\theta_{1p}$ See Page

DETERMINATION OF V_o, H_o and θ AT NEUTRAL POINT
FOR NON-PRISMATIC ARCH (Cont.)

X_{1p}	$\bar{y}(\frac{S^2}{8}+\frac{X_{1p}^2}{2})$	$R^3/3 \cos^3\theta_{1p}$	$\frac{R^2}{2}X_{1p}$	$\theta_{1p}+\sin\theta_{1p}\cos\theta_{1p}$	$\frac{(R-r)^3}{3}$
0	.08282799	.12708604	0	0	.04825437
.1	.08614110	.12341765	.02628125	.27498	.04825437
.2	.09608046	.11280648	.05256250	.54464	.04825437
.3	.11264606	.09583995	.07884375	.80332	.04825437
.4	.13583790	.07369496	.10512500	1.04458	.04825437

Load Point	ΔH	δ_{hh}	H_o
1	.00403632	.0032212	1.2530465
2	.00373100	.0032212	1.1582640
3	.00288071	.0032212	.8942971
4	.00170372	.0032212	.5289064
5	.00056584	.0032212	.1756612

From Formula Previously Derived

$$\theta = \frac{S^2}{8} + \frac{(X_{1p})^2}{2} - \frac{X_{1p}S}{2} \qquad M_o = -\frac{\theta}{\alpha_{\theta m}}$$

Load Point	X_{1p}	$S^2/8$	$\frac{(X_{1p})^2}{2}$	$\frac{X_{1p}S}{2}$	θ	M_o
1	0	.125	-	0	.125	.125
2	.1	.125	.005	.05	.080	.080
3	.2	.125	.020	.10	.045	.045
4	.3	.125	.045	.15	.020	.020
5	.4	.125	.080	.20	.005	.005

DETERMINATION OF FIXED END MOMENT FOR NON-PRISMATIC ARCH

$$M_l^f = V_o \, {}^S/_2 - H_o \left[r-(R-\bar{y})\right] + M_o$$

$$M_r^f = V_o \, {}^S/_2 - H_o \left[r-(R-\bar{y})\right] + M_o + 1 \left({}^S/_2 - X_{1p}\right)$$

X_{1p}	$V_o \, {}^S/_2$	$H_o \left[r-(R-\bar{y})\right]$	M_o
0	(-.5000000)(-.5)	(-1.2530485)(-.1376239)	-.1250000
.1	(-.3620000)(-.5)	(-1.1582640)(-.1376239)	-.0850000
.2	(-.2160000)(-.5)	(- .8942971)(-.1376239)	-.0450000
.3	(-.1040000)(-.5)	(- .5289084)(-.1376239)	-.0200000
.4	(-.0280000)(-.5)	(-.1756612)(-.1376239)	-.0050000

M_l^f	M_r^f
-.047449	-.047449
-.063405	-.015405
-.060077	+.023923
-.040790	+.055210
-.015175	+.056825

The Fixed End Moments shown above are caused by 1 kip loads placed at distances X_{1p} to the right of the crown. F.E.M. (left) as caused by a 1^k load to the right of the crown equals F.E.M. (right) as caused by a 1^k load placed at the corresponding point to the left of the crown. All F.E.M. values were tabulated on this basis.

DETERMINATION OF CARRY OVER FACTOR, THRUST, AND ABSOLUTE STIFFNESS (NON-PRISMATIC)

Span Length = 1.0 Rise/Span = .20 R = .725' $\frown{L}$ = 1.103479

Carry Over

(1) $EI_c \delta'_{vm} = EI_c \alpha'_{mv} = \frac{S^2}{2} = .5000$

(2) $EI_c \delta'_{hm} = EI_c \alpha'_{mh} = \frac{1}{2}\left[\hat{L}R - S(R-r)\right] = \frac{1}{2}\left[(1.103479)(.725)-(1)(.525)\right]$

$= .13762388$

(3) $EI_c \alpha_{mm} = S = 1.000$

(4) $EI_c \delta_{vv} = \frac{S^3}{3} = .33333$

(5) $EI_c \delta_{hh} = SR^2 - \frac{S^3}{12} - \hat{L}R(R-r) = .52562500 - .08333333 - .42013007$

$= .022161600$

(6) $EI_c \delta_{vh} = EI_c \delta_{hv} = \frac{S}{4}\left[\hat{L}R - S(R-r)\right] = \frac{1}{4}\left[(1.103479)(.725)-(1)(.525)\right]$

$= .06881194$

$$D_2' = \frac{(EI_c)^2\left[(\delta'_{vm})(\delta_{hh}) - (\delta'_{hm})(\delta_{vh})\right]}{(EI_c)^2\left[(\delta_{hv})(\delta_{vh}) - (\delta_{vv})(\delta_{hh})\right]} = \frac{(1)(5) - (2)(6)}{(6)(6) - (4)(5)}$$

$$D_2' = \frac{.01108080 - .00947017}{.00473508 - .00738720} = .6073012$$

$V_a = D_2' M_a = -.6073012 M_a$

$-M_b = -(1+D_2')(S)M_a = -(1-.6073012)(1)M_a = -.3926988\ M_a$

Thrust

$$D_1' = \frac{(EI_c)^2\left[(\delta'_{vm})(\delta_{hv}) - (\delta'_{hm})(\delta_{vv})\right]}{(EI_c)^2\ (\delta_{hh})(\delta_{vv}) - (\delta_{hv})(\delta_{vh})]} = \frac{(1)(6) - (2)(4)}{(5)(4) - (6)(6)}$$

$$= \frac{.03440597 - .04587462}{.00738720 - .00473508} = -4.324337$$

$H_a = D_1' M_a = -4.324337 M_a$

DETERMINATION OF CARRY OVER FACTOR, THRUST, AND ABSOLUTE STIFFNESS (NON-PRISMATIC)

Absolute Stiffness

From Formulas Previously Derived

When $\theta_4 = 1$ $\quad M_a = \frac{EI_c}{D_3}$ where $D_3 = (\alpha_{am} + D_2'\alpha'_{av} + D_1'\alpha'_{ah})EI_c$

$(EI_c)(\alpha_{am}) =$ [illegible] $= 1.000$

$(EI_c)(D_2')(\alpha'_{av}) = (-.60730120)(.5) = -.30365060$

$(EI_c)(D_2')(\alpha'_{ah}) = (-4.324337)(.13762388) = -.59513204$

$D_3' = 1.000 - .30365060 - .59513204 = +.10121736$

Absolute Stiffness $= M_a = \frac{EI_c}{D_3'} = 9.8797281\ EI_c$

DETERMINATION OF MOMENT AND REACTIONS DUE TO THE EFFECT OF SPREAD (NON-PRISMATIC)

Moment

From Formulas Previously Derived

$$M_a = \frac{EI_c \Delta H}{D_4} \quad \text{where } D_4' = \frac{(EI_c\,\delta_{vh})(EI_c\,\delta'_{hm}) - (\delta'_{vm})(\delta_{hh})(EI_c)^2}{(EI_c)(\delta_{vh})}$$

(1) $(EI_c\,\delta'_{hm})(EI_c\,\delta_{vh}) = (.13762388)(.06881194) = .00947017$

(2) $(EI_c\,\delta'_{vm})(EI_c\,\delta_{hh}) = (.5)(.02216160) = .0110808$

(3) $(EI_c\,\delta_{vh}) = .06881194$

(4) $D_4' = \frac{(1)-(2)}{(3)} = \frac{.00947017 - .01108080}{.06881194} = -.02340631$

(5) $M_a = \frac{EI_c \Delta H}{D_4'} = \frac{EI_c \Delta H}{-.02340631} = -42.723521\ EI_c \Delta H$

Vertical Reaction

From Formulas Previously Derived

$V_a = 0$

Horizontal Reaction

From Formulas Previously Derived

$$H_a = \frac{EI_c \Delta H}{D_5'} \quad \text{where } D_5' = \frac{(EI_c\,\delta'_{vm})(EI_c\,\delta_{hh}) - (EI_c\,\delta'_{hm})(EI_c\,\delta_{vh})}{EI_c\,\delta'_{vm}}$$

(1) $(EI_c\,\delta'_{vm})(EI_c\,\delta_{hh}) = (.5)(.02216160) = .0110808$

(2) $(EI_c\,\delta'_{hm})(EI_c\,\delta_{vh}) = (.13762388)(.06881194) = .00947017$

(3) $EI_c\,\delta'_{vm} = .5$

(4) $D_5' = \frac{(1)-(2)}{(3)} = \frac{.0110808 - .00947017}{.5} = +.00322127$

(5) $H_a = \frac{EI_c \Delta H}{D_5'} = \frac{EI_c \Delta H}{.00322127} = (+)310.43875\ EI_c \Delta H$

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